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# TIME DOMAIN FUNDAMENTAL SOLUTION TO BIOT'S COMPLETE EQUATIONS OF DYNAMIC POROELASTICITY. PART I: TWO-DIMENSIONAL SOLUTION†

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**Abstract**—This paper develops transient fundamental solution for Biot's full dynamic two-dimensional equations of poroelasticity. An explicit, well-posed Laplace transform domain fundamental solution is obtained for the governing differential equations which are established in terms of solid displacements and fluid pressure. In some limiting cases, the solutions are shown to reduce to those of classical elastodynamics and steady state poroelasticity, thus ensuring the validity of our result. The closed-form transient fundamental solutions both for the limiting case (early time approximation) and for the general case are derived from corresponding ones in the Laplace transform domain by means of Laplace transform techniques. They represent the very first satisfactory fundamental solutions in the real-time domain for full dynamic poroelasticity. Some characteristics of the solutions are investigated. Selected numerical results are presented to demonstrate the features of the waves, and the accuracy of the solution is established by comparing them with Laplace transform domain solutions. To complete the ingredients for developing the BEM, the Betti's reciprocal theorem is extended to the full dynamic poroelasticity, based on which a time domain boundary integral equation is obtained.

## INTRODUCTION

The study of the mechanics of fluid saturated porous elastic media has a long and interesting history (Telford, 1821; Delesse, 1848; Darcy, 1856; Fillunger, 1913). However, the theoretical basis was not established until the early twenties of this century, when Terzaghi (1923) deduced on a somewhat intuitive basis his famous differential equations for the description of one-dimensional consolidation and settlement process in clay beds. In the Terzaghi theory the porous-media flow is uncoupled from the process of deformation and the governing equation is reduced to a diffusion equation.

Extending the work of Terzaghi (1923), Biot (1941a, 1955, 1956a) published a series of papers dealing with a general theory of behavior of what is now termed poroelastic materials. Biot's poroelastic materials are two-phase solid–fluid-filled systems. The porous solid skeleton is linearly elastic and undergoes small deformations while flow of the fluid produced by deformation of the material is governed by Darcy's law. It has been shown (Schiffman *et al.*, 1969) that Biot (1941a, 1955) more adequately described the soil consolidation behaviour than Terzaghi by providing a coupling between the solid and fluid stresses and strains. Later Biot (1956b, c, 1962), using a phenomenological approach, developed a full dynamic theory for study of wave propagations in fluid saturated porous media. His pioneering work marked the beginning of an era of study of dynamic poroelasticity problems. Biot's contribution to the development of the theory of poroelasticity has been summarized in survey papers by Paria (1963) and de Boer and Ehlers (1988).

At a later date the essential correctness of Biot theory has been confirmed both from a two-scaled analysis of the Navier–Stokes equations (Burridge and Keller, 1981) and from the viewpoint of the theory of mixtures (Bowen, 1976, 1982; Prevost, 1980), the general theoretical framework of which was first developed by Truesdell and Toupin (1960) and later developed by Green and Naghdi (1965); however, some of these more general theories often contain constitutive constants that are intractable in practice.

The theory has also been firmly establisihed by the experimental observations. The

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existence of a strongly attenuated slow compressional wave, which has been predicted by Biot, was observed in water-saturated fused glass bead samples (Plona, 1980), in the artificial porous media (Plona and Johnson, 1984) and in human bones (Lakes *et al.*, 1983). Excellent agreements between these observations and the Biot theory are reported. On the other hand, the physical significance of the material coefficients involved in Biot's theory has been well explored, e.g. by Biot and Willis (1957), Fatt (1959), Scholz *et al.* (1973), and Rice and Cleary (1976), and measured in the laboratory by Yew and Jogi (1978).

A number of investigators have succeeded in obtaining analytical solutions to the Biot theory via classical mathematical analyses. However, due to the fact that the consideration of viscous coupling and inertia effects presents formidable difficulties in the solution process, existing studies are concerned mainly with frequency domain problems, quasistatic problems, or dynamic problems with viscous dissipation being neglected. Among them, Biot himself provided the earliest solutions to his equations (Biot, 1941a, b; Biot and Clingan, 1941, 1942). Few ingenious analytical solutions were those contributed by McNamee and Gibson (1960), Rice and Cleary (1976) and Mei and Fonda (1981). Other later investigations of note were those by Shanker *et al.* (1978), Gibson and McNamee (1963), Gibson *et al.* (1970), Garg *et al.* (1974), Booker (1974), Booker and Carter (1986), Simon *et al.* (1984) and Halpern and Christiano (1986). On the other hand the partial list of authors, who have treated the subject of the characteristics of the wave in poroelastic media, includes Biot (1956b, c) himself, Hardin (1961), Geertsman and Smit (1961), Ishihara (1967), Hsieh and Yew (1973), Yew and Jogi (1976), Deresiewicz *et al.* (1960–1967), Jones (1961), Gardner (1962), Yamamoto (1977) and Yamamoto *et al.* (1978).

Historically, classical mathematics was the only effective avenue of solving governing differential equations for description of practical engineering problems. Fortunately with the advent of high-speed digital computers, more and more engineering analyses are performed via computational methods (FEM, BEM, FDM), which have been expanding at an increasing rate since the 1960s. However it needs to be emphasized, even at the risk of appearing trivial and repetitive, that computational methods can, and in many cases must, benefit from classical mathematical analysis. This is true especially in the case of the Boundary Element Method where a specific and important subject is to determine the fundamental solutions and boundary integral equations pertaining to original governing differential equations via classical mathematics. With the recent growing interest in the boundary element method and its application to the various branches of applied mechanics, there is a growing need for the determination of transient fundamental solutions for Biot's full dynamic theory. In addition to their attributes in BEM, applied mechanics and applied mathematics, the fundamental solution also offers important physical insights which are invaluable to the development of solutions of practical engineering problems encountered in geomechanics, seismology, oil exploration, earthquake engineering and biomechanics.

Fundamental solutions for the corresponding quasistatic problem were derived by Cleary (1977) following from the earlier work of Nowacki (1966), while closed-form Laplace domain quasistatic poroelastic fundamental solutions were obtained by Cheng and Liggett (1984a, b). However, it seems that the first attempt to obtain fundamental solutions for dynamic poroelasticity was made by Burridge and Vargas (1979), who, in addition to presenting a general solution procedure similar to that of Deresiewicz (1960), used the saddle point method to obtain displacements at large distances due to a point force in the solid. Later, Norris (1985) derived time harmonic Green functions for a point force in the solid and a point force in the fluid. He also obtained explicit asymptotic approximations for far-field displacements, as well as those for high and low frequency responses. More recently, Kaynia and Banerjee (1992) used a solution scheme similar to that of Norris (1985) and derived the fundamental solution in the Laplace transform domain as well as transient short-time solution. Unfortunately, the Burridge and Vargas solution was obtained for three forces which is not complete enough, while the Norris and Kaynia and Banerjee solutions were sought for six variables $(u, w)$ and six forces $(f_1, f_2)$, which is too much. The well-known time-harmonic poroelastic fundamental solutions were given by Bonnet (1987) and Boutin *et al.* (1987). But they are not without drawbacks. The errors in Bonnet's paper have been shown by Dominguez (1991, 1992). Additionally, Bonnet's solution does not allow clear identification of the sources involved in the calculation, as was pointed out by Boutin *et al.* (1987). Boutin on the other hand worked on the equations which are based upon the homogenization theory for periodic structures (Auriault, 1980; Auriault *et al.*, 1985). However, in the generalized Darcy's law of Boutin's work, the effect of fluid acceleration term is neglected. Therefore the fundamental solutions fail to be able to be applied to Biot's full dynamic equations. It should also be mentioned that Boutin's paper contained some inaccuracies: due to the omission of the multiplier $i\omega$ for all terms in eqn (4) of Section 1.2 (Boutin, 1987) when the harmonic fluid source is added, the corresponding differential operator matrix and final fundamental solution matrix reduces to a symmetrical form. However, they should not be symmetrical as $G_{i4}$ and $G_{4i}$ cannot be equal. Recently Wiebe and Antes (1991) obtained a time domain fundamental solution for the Biot (1956b) type dynamic poroelasticity by neglecting the viscous coupling (the friction between fluid particles and the porous elastic structure) and without numerical evaluation of the kernel functions. Therefore time domain Green's functions for full dynamic poroelasticity still remain unsolved.

In the present paper, the author succeeds in providing for the first time an analytical time domain Green's function for two-dimensional full dynamic poroelasticity. Initially, governing equations are established in terms of solid displacement and excessive fluid pressure, i.e. $u-p$ model, and an explicit Laplace-transform domain fundamental solution is obtained in closed form and verified by checking limiting cases. Thereafter, the transient fundamental solutions both for the limiting case and for the general case are derived via Laplace-transform techniques. Some characteristics of the analytical solution are discussed. Numerical results are plotted to portray the features of the waves and the accuracy of the solution is established by comparing them with the corresponding Laplace-transform domain solutions. Finally the Betti's reciprocal theorem is extended to the dynamic poroelasticity and the time domain boundary integral equation is derived, thus opening up a road for implementation of time domain BEM to solve full dynamic poroelastic problems.

## GOVERNING EQUATIONS

The elegantly extended formulation of the equations governing the transient response of poroelastic media, which were first established by Biot (1956b, c), has been given by Zienkiewicz *et al.* (1980) and Zienkiewicz and Shiomi (1984), and this treatment will be followed here.

A first constitutive relation for the total stress $\sigma_{ij}$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + \mu_{j,i}) - \alpha p \delta_{ij}. \tag{1}$$

A second constitutive relation is needed for the alteration in $\theta$, which represents the increment of fluid content.

$$\theta = \alpha u_{k,k} + \frac{1}{Q} p, \tag{2}$$

where $u_i$ is the displacement of the solid skeleton, $p$ denotes the fluid pressure, the elastic constants $\lambda$ and $\mu$ are termed drained Lamé constants, $\alpha$ and $Q$ are dimensionless material parameters which describe relative compressibilities of the constituents and are defined as $\alpha = 1-(K_D/K_S)$, $(1/Q) = (n/K_f)+((\alpha-n)/K_S)$, where $K_D$, $K_f$ are the bulk modulus of drained solid skeleton and the fluid while $K_S$ is that relating the pore fluid pressure to the volumetric strain in the solid skeleton. $n$ is porosity and $\delta_{ij}$ is Kronecker's delta.

Next, the generalized Darcy's law is invoked to specify the equilibrium of the fluid phase:

$$\dot{w}_i = -\kappa(p_{,i} + \rho_f \ddot{u}_i + m\ddot{w}_i) \tag{3}$$

where $w_i$ denotes the average displacements of the fluid relative to the solid, i.e. $w_i = n(U_i - u_i)$ where $U_i$ is the average displacement of the fluid, $\dot{w}_i = (\mathrm{d}w_i/\mathrm{d}t)$ is the average relative velocity of seepage measured over the total area, $\kappa = k/\eta$ is the permeability coefficient, with $\eta$ and $k$ denoting the fluid dynamic viscosity and the intrinsic permeability of the solid skeleton respectively, $\rho_f \ddot{u}_i$ and $m\ddot{w}_i$ are the components of the perturbation of inertia force acting on the fluid, $\rho_f$ is fluid density and $m = \rho_f/n$ (Zienkiewicz *et al.*, 1980) or $m = (\rho_a/n^2) + (\rho_f/n)$ (Biot, 1956b; Bonnet and Auriault, 1985; Cheng and Badmus, 1991), where $\rho_a$ is the apparent mass density corresponding to the work done by the solid phase to the fluid phase due to the relative motion between them.

The fluid mass flux vector $q_i$ is defined in the following relation:

$$q_i = \dot{w}_i. \tag{4}$$

Now an equilibrium equation is introduced:

$$\sigma_{ij,j} + f_i = \rho\ddot{u}_i + \rho_f \ddot{w}_i, \tag{5}$$

in which $\rho = (1-n)\rho_s + n\rho_f$ is the density of the solid-fluid mixture, $\rho_s$ and $f_i$ are the density of the solid material and the body force, respectively.

The equation set is completed by a continuity of flow condition which concerns the rate of fluid accumulation as:

$$\dot{\theta} + q_{i,i} = \gamma, \tag{6}$$

where $\gamma$ is the rate of fluid injection into the media. Introducing eqns (2) and (4) into (6) leads to the following expression for the continuity equation:

$$\dot{w}_{k,k} = -\alpha\dot{u}_{k,k} - \frac{1}{Q}\dot{p} + \gamma. \tag{7}$$

Taking the Laplace transform of eqns (1), (3), (5), (7), assuming that the initial conditions are zero and performing appropriate substitution, one obtains a new form of the equations in the Laplace transform space with solid displacement $\tilde{u}_i$ and the pressure $\tilde{p}$ as four independent variables to describe the behavior of the porous medium. That is:

$$(\lambda + \mu)\tilde{u}_{j,ij} + \mu\tilde{u}_{i,jj} - \alpha_1\tilde{p}_{,i} - \rho_1 s^2\tilde{u}_i + \tilde{f}_i = 0, \tag{8}$$

$$\zeta\tilde{p}_{,ii} - \frac{s}{Q}\tilde{p} - \alpha_2 s\tilde{u}_{i,i} + \tilde{\gamma} = 0, \tag{9}$$

in which $i, j = 1, 2, 3$, the transformed function is defined by:

$$\tilde{f}(x, s) = \int_0^{\infty} f(x, t)e^{-st}\,\mathrm{d}t, \tag{10}$$

where $s$ is the Laplace transform parameter and the tilde denotes the Laplace transformation, $\zeta = ((1/\kappa) + ms)^{-1}$, $\alpha_1 = \alpha - \rho_f s\zeta$, $\alpha_2 = \alpha - \rho_f s\zeta$, $\rho_1 = \rho - \rho_f^2 s\zeta$. Notice that separate parameters $\alpha_1$ and $\alpha_2$ have been introduced in eqns (8) and (9) to represent the same function (i.e. $\alpha - \rho_f s\zeta$). This is necessary in order to extend the work to dynamic thermoelasticity (Chen, 1992, 1993a, b).

The two-dimensional governing equations can be derived directly from eqns (8) and (9) by assuming that displacement components are independent of one of the coordinates, say $x_3$ and by setting $u_3 = 0$ and $\partial/\partial x_3 = 0$. Thus it leads to the same form as (8), (9) except for the indices $i, j = 1, 2$.

Nondimensionalized quantities are introduced to present the result in a meaningful manner (Ghabboussi and Wilson, 1972; Simon *et al.*, 1984). Thus we define dimensionless coordinates and time by means of:

$$\xi_i = \frac{x_i}{\rho\kappa C_P} \quad \text{and} \quad \tau = \frac{t}{\rho\kappa}, \tag{11}$$

where $C_P$ is the propagation speed associated with waves moving through the porous media without relative motion between the fluid and the solid phase given as:

$$C_P = \sqrt{\frac{\lambda+2\mu+\alpha^2 Q}{\rho}}. \tag{12}$$

Next, we define a dimensionless displacement and pore pressure through:

$$U_i = \frac{u_i}{\rho\kappa C_P}, \quad P = \frac{p}{\rho C_P^2}, \tag{13}$$

and denote:

$$\lambda^* = \frac{\lambda}{\lambda+2\mu+\alpha^2 Q}, \tag{14a}$$

$$\mu^* = \frac{\mu}{\lambda+2\mu+\alpha^2 Q}, \tag{14b}$$

$$Q^* = \frac{Q}{\lambda+2\mu+\alpha^2 Q}, \tag{14c}$$

$$\rho^* = 1, \tag{14d}$$

$$\rho_f^* = \frac{\rho_f}{\rho}, \tag{14e}$$

$$m^* = \frac{m}{\rho}, \tag{14f}$$

$$\kappa^* = 1. \tag{14g}$$

The nondimensional form for eqns (8) and (9) is then:

$$(\lambda^*+\mu^*)\tilde{U}_{j,ij}+\mu^*\tilde{U}_{i,jj}-\alpha_1^*\tilde{P}_{,i}-\rho_1^* s^2\tilde{U}_i+\tilde{F}_i = 0, \tag{15}$$

$$\zeta^*\tilde{P}_{,ii}-\frac{s}{Q^*}\tilde{P}-\alpha_2^* s\tilde{U}_{i,i}+\tilde{\Gamma} = 0 \tag{16}$$

where $\alpha_1^*, \alpha_2^*, \rho_1^*, \zeta^*$ are defined by:

$$\alpha_1^* = \alpha_2^* = \alpha-\rho_f^* s\zeta^*,$$

$$\rho_1^* = \rho^*-(\rho_f^*)^2 s\zeta^*,$$

$$\zeta^* = \frac{1}{\dfrac{1}{\kappa^*}+m^* s}. \tag{17}$$

For three-dimensional $i, j = 1,2,3$ and two-dimensional $i, j = 1,2$.

The nondimensional form derived above is attractive from the pure mathematical point of view and makes the mathematics a little more concise and elegant.

## LAPLACE TRANSFORM DOMAIN FUNDAMENTAL SOLUTION

Fundamental solutions are the response of the medium to point excitations which is a Dirac delta function in space, i.e. $\delta(x_i)$ and either a Dirac delta function or a Heaviside step function i.e. $H(t)$ in time. However, for pedagogical purposes and its future application in BEM it is better to consider the solution which results from a source which is a Heaviside step function in time. For a continuous unit line force in the $i$-th direction suddenly applied at the origin, i.e. $f_i(x,t) = \delta(x_1)\delta(x_2)H(t)$, and a unit rate of fluid line injection at the origin, i.e. $\gamma(x,t) = \delta(x_1)\delta(x_2)H(t)$, the Laplace transform of which is $(1/s)\delta(x_1)\delta(x_2)$. Now following the Hörmander's Method or Kupradze's Method, which has been thoroughly expounded in the classical works by Hörmander (1964) and Kupradze (1979), the two-dimensional Laplace domain fundamental solutions are to be derived with only the essential steps of the method applied to the present problem being listed.

It is convenient to write the basic eqns (8) and (9) for the two-dimensional case in their matrix form as:

$$\mathbf{B}(\partial\mathbf{x},s) = [B_{mn}(\partial\mathbf{x},s)]_{3\times 3}, \tag{18}$$

where

$$B_{ij}(\partial\mathbf{x},s) = (\lambda+\mu)\frac{\partial^2}{\partial x_i\,\partial x_j} + \delta_{ij}(\mu\Delta - \rho_1 s^2), \tag{19a}$$

$$B_{i3}(\partial\mathbf{x},s) = -\alpha_1\frac{\partial}{\partial x_i}, \tag{19b}$$

$$B_{3j}(\partial\mathbf{x},s) = -\alpha_2 s\frac{\partial}{\partial x_j}, \tag{19c}$$

$$B_{33}(\partial\mathbf{x},s) = \zeta\Delta - \frac{s}{Q}, \tag{19d}$$

where $m,n = 1,2,3$; $i,j = 1,2$; $\Delta$ is the Laplacian operator. Now the eqns (8) and (9) are rewritten in the form:

$$\mathbf{B}(\partial\mathbf{x},s)\tilde{\mathbf{U}} + \tilde{\mathbf{F}} = 0, \tag{20}$$

where $\tilde{\mathbf{U}} = (\tilde{U}_n)$ and $\tilde{\mathbf{F}} = (\tilde{F}_n)$ are three-component vectors, i.e. $\tilde{U}_i = \tilde{u}_i$, $\tilde{U}_3 = \tilde{p}$, $\tilde{F}_n = (1/s)\delta(x_1)\delta(x_2)$. Note that $\mathbf{B}(\partial\mathbf{x},s)$ matrix is not symmetrical. The problem is to find the fundamental solution matrix $\tilde{\mathbf{G}} = [\tilde{G}_{ij}]_{3\times 3}$, which satisfies:

$$\mathbf{B}(\partial\mathbf{x},s)\tilde{\mathbf{G}} + \mathbf{I}\frac{1}{s}\delta(\mathbf{x}) = 0, \tag{21}$$

where $\mathbf{I}$ is the unit matrix. We now proceed to solve eqn (21); by calculating the cofactor of element $B^*_{ij}(\partial\mathbf{x},s)$ in $B_{ij}(\partial\mathbf{x},s)$ we find:

$$B^*_{ij} = -\frac{\partial^2}{\partial x_i\,\partial x_j}\left[(\lambda+\mu)\left(\zeta\Delta - \frac{s}{Q}\right) - \alpha_1\alpha_2 s\right] - \delta_{ij}(\lambda+2\mu)\zeta(\Delta-\lambda_1^2)(\Delta-\lambda_2^2),$$

$$B^*_{i3} = \mu\alpha_1\frac{\partial}{\partial x_i}(\Delta-\lambda_3^2),$$

$$B^*_{3i} = \mu s\alpha_2\frac{\partial}{\partial x_i}(\Delta-\lambda_3^2),$$

$$B^*_{33} = \mu(\lambda+2\mu)(\Delta-\lambda_3^2)\left(\Delta - \frac{\mu}{\lambda+2\mu}\lambda_3^2\right). \tag{22}$$

Considering the differential operator $\mathbf{B}^*(\partial\mathbf{x}, s)$ built from the cofactors of $\mathbf{B}(\partial\mathbf{x}, s)$, we have:

$$B_{ik}(\partial\mathbf{x}, s)B^*_{kj}(\partial\mathbf{x}, s) = \delta_{ij}\det(\mathbf{B}(\partial\mathbf{x}, s)). \tag{23}$$

Let us now assume that $\varphi$ is the scalar solution to the equation

$$\det(\mathbf{B}(\partial\mathbf{x}, s))\varphi + \frac{1}{s}\delta(\mathbf{x}) = 0, \tag{24}$$

and that $\varphi$ satisfies also the condition

$$[\det(\mathbf{B}(\partial\mathbf{x}, s))\mathbf{I}]\varphi + \frac{1}{s}\mathbf{I}\delta(\mathbf{x}) = 0, \tag{25}$$

which gives

$$\mathbf{B}(\partial\mathbf{x}, s)\mathbf{B}^*(\partial\mathbf{x}, s)\varphi + \mathbf{I}\frac{1}{s}\delta(\mathbf{x}) = 0. \tag{26}$$

Consequently, in comparison with eqn (21) we get:

$$\tilde{\mathbf{G}} = \mathbf{B}^*(\partial\mathbf{x}, s)\varphi. \tag{27}$$

Equation (27) enables us to determine the nine functions $\tilde{G}_{ij}$ by applying the differential operator $\mathbf{B}^*(\partial\mathbf{x}, s)$ to the single unknown function $\varphi$, the computation of $\det(\mathbf{B}(\partial\mathbf{x}, s))$ leads to:

$$\det(\mathbf{B}(\partial\mathbf{x}, s)) = \mu(\lambda+2\mu)\zeta(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2), \tag{28}$$

where

$$\lambda_1^2+\lambda_2^2 = \Lambda^2+\frac{s}{Q\zeta}+\frac{\alpha_1\alpha_2 s}{\zeta(\lambda+2\mu)}, \tag{29a}$$

$$\lambda_1^2\lambda_2^2 = \Lambda^2\frac{s}{Q\zeta}, \tag{29b}$$

$$\lambda_3^2 = \frac{\rho_1 s^2}{\mu}, \tag{29c}$$

$$\Lambda^2 = \frac{\rho_1 s^2}{\lambda+2\mu}, \tag{29d}$$

$$\zeta = \left(\frac{1}{\kappa}+ms\right)^{-1}, \tag{29e}$$

$$\alpha_1 = \alpha_2 = \alpha-\rho_f s\zeta, \tag{29f}$$

$$\rho_1 = \rho-\rho_f^2 s\zeta. \tag{29g}$$

Considering eqns (29a), (29b), $\lambda_1^2$ and $\lambda_2^2$ are given as the roots of the following characteristic equation:

$$(\lambda_i^2)^2-\left(\Lambda^2+\frac{s}{Q\zeta}\left(1+\frac{Q\alpha_1\alpha_2}{\lambda+2\mu}\right)\right)\lambda_i^2+\Lambda^2\frac{s}{Q\zeta} = 0. \tag{30}$$

Substituting eqns (29d–g) into (30) and multiplying both sides by $Q(\lambda+2\mu)/\rho^2$, after algebra, one obtains:

$$(C_P^2\lambda_i^2-s^2)(C_P^2Q^*\lambda_i^2-m^*s^2)-(C_P^2\alpha Q^*\lambda_i^2-\rho_f^*s^2)^2-\frac{1}{\rho\kappa}s(C_P^2\lambda_i^2-s^2)=0, \quad (31)$$

where $\rho_f^*$, $m^*$, $Q^*$. $C_P$ are defined in eqns (12) and (14). Equation (31) can also be written as:

$$(C_P^4Q^*-C_P^4\alpha^2(Q^*)^2)\lambda_i^4-(C_P^2m^*+C_P^2Q^*-2\alpha Q^*\rho_f^*C_P^2)s^2\lambda_i^2$$
$$+(m^*-(\rho_f^*)^2)s^4-\frac{1}{\rho\kappa}s(C_P^2\lambda_i^2-s^2)=0. \quad (32)$$

Let us recast eqn (32) as:

$$C_1^2C_2^2\lambda_i^4-(C_1^2+C_2^2)s^2\lambda_i^2+s^4-\delta s(C_P^2\lambda_i^2-s^2)=0. \quad (33)$$

When comparing (32) with (33) we get:

$$C_1^2+C_2^2=\frac{C_P^2m^*+C_P^2Q^*-2\alpha Q^*\rho_f^*C_P^2}{m^*-(\rho_f^*)^2}, \quad (34a)$$

$$C_1^2C_2^2=\frac{C_P^4Q^*-C_P^4\alpha^2(Q^*)^2}{m^*-(\rho_f^*)^2}, \quad (34b)$$

$$\delta=\frac{1}{\rho\kappa(m^*-(\rho_f^*)^2)}. \quad (34c)$$

Consequently $C_1^2$, $C_2^2$ are given by:

$$C_{1,2}^2=\frac{(m^*+Q^*-2\alpha Q^*\rho_f^*)\pm\sqrt{(m^*+Q^*-2\alpha Q^*\rho_f^*)^2-4(m^*-(\rho_f^*)^2)(Q^*-\alpha^2(Q^*)^2)}}{2(m^*-(\rho_f^*)^2)}C_P^2, \quad (35)$$

where $C_1$ and $C_2$ are introduced to express the velocities of the first and second longitudinal waves of the associated nondissipative (drained) case. $\delta$ is the poromechanical coupling parameter.

Similarly eqn (29c) can be written as:

$$\lambda_3^2=\frac{s^2}{C_S^2}\frac{(m^*-(\rho_f^*)^2)s+(m^*-(\rho_f^*)^2)\delta}{m^*s+(m^*-(\rho_f^*)^2)\delta} \quad (36)$$

where $C_S^2=\mu/\rho$ is the drained shear wave velocity.

For zero values of viscous coupling (i.e. $\delta=0$). Equation (36) yields:

$$\lambda_3^2=\frac{s^2}{C_3^2}, \quad (37a)$$

$$C_3=\left(\frac{m^*}{m^*-(\rho_f^*)^2}\right)^{1/2}C_S. \quad (37b)$$

$C_3$ is the velocity of the transverse waves of the associated nondissipative case. Equation

(24) and (28) may now be combined to yield:

$$\mu(\lambda+2\mu)\zeta(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)s\varphi+\delta(\mathbf{x}) = 0. \tag{38}$$

Now we have to find the solution $\varphi$ of eqn (38), and for the function $\Phi = \mu(\lambda+2\mu)s\zeta\varphi$, we have the equation:

$$(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)\Phi+\delta(\mathbf{x}) = 0. \tag{39}$$

To solve eqn (39), we assume:

$$\varphi_1 = (\Delta-\lambda_2^2)(\Delta-\lambda_3^2)\Phi, \tag{40a}$$

$$\varphi_2 = (\Delta-\lambda_3^2)(\Delta-\lambda_1^2)\Phi, \tag{40b}$$

$$\varphi_3 = (\Delta-\lambda_1^2)(\Delta-\lambda_2^2)\Phi. \tag{40c}$$

Substituting eqns (40a–c) into eqn (39), respectively we get:

$$(\Delta-\lambda_i^2)\varphi_i = -\delta(\mathbf{x}) \quad i = 2,3,3. \tag{41}$$

This is the Helmholtz equation in $E_2$, the solutions of which, corresponding to three waves, i.e. diffusive wave (slow compressional wave), pressure wave (fast compressional wave) and shear wave (equivoluminal wave), are well-known functions:

$$\varphi_i = \frac{1}{2\pi}K_0(\lambda_i r) \quad i = 1,2,3, \tag{42}$$

where $K_0$ is the modified Bessel function of zero order.

In eqns (40a–c), $\varphi_2$ subtracted from $\varphi_1$ and $\varphi_3$ subtracted from $\varphi_2$ leaves:

$$\Delta\Phi-\lambda_3^2\Phi = \frac{\varphi_1-\varphi_2}{\lambda_1^2-\lambda_2^2}, \tag{43a}$$

$$\Delta\Phi-\lambda_1^2\Phi = \frac{\varphi_2-\varphi_3}{\lambda_2^2-\lambda_3^2}. \tag{43b}$$

Equation (43b) subtracted from eqn (43a) leaves:

$$\Phi = \frac{1}{\lambda_1^2-\lambda_3^2}\left(\frac{\varphi_1-\varphi_2}{\lambda_1^2-\lambda_2^2} - \frac{\varphi_2-\varphi_3}{\lambda_2^2-\lambda_3^2}\right). \tag{44}$$

Substituting eqn (42) into eqn (44) we obtain, after computation:

$$\Phi = \frac{1}{2\pi}\sum_{i=1}^{3}\frac{1}{(\lambda_{i+1}^2-\lambda_i^2)(\lambda_{i+2}^2-\lambda_i^2)}K_0(\lambda_i r) \tag{45}$$

with $\lambda_4 = \lambda_1$ and $\lambda_5 = \lambda_2$.

Thus:

$$\varphi(r,s) = \frac{1}{2\pi\mu(\lambda+2\mu)\zeta s}\sum_{i=1}^{3}\frac{1}{(\lambda_{i+1}^2-\lambda_i^2)(\lambda_{i+2}^2-\lambda_i^2)}K_0(\lambda_i r). \tag{46}$$

By applying the operator $\mathbf{B}^*_{ij}(\partial\mathbf{x},s)$ to $\varphi(r,s)$, we get the Green matrix in the two-dimensional case as:

$$\tilde{G}_{ij} = -(A_{ij}\lambda_1 K_1(\lambda_1 r) + B_{ij}\lambda_1^2 K_0(\lambda_1 r))\frac{\Lambda^2-\lambda_2^2}{\lambda_2^2-\lambda_1^2}\frac{1}{\rho_1 s^3} + (A_{ij}\lambda_2 K_1(\lambda_2 r) + B_{ij}\lambda_2^2 K_0(\lambda_2 r))$$
$$\times\frac{\Lambda^2-\lambda_1^2}{\lambda_2^2-\lambda_1^2}\frac{1}{\rho_1 s^3} - (A_{ij}\lambda_3 K_1(\lambda_3 r) + B_{ij}\lambda_3^2 K_0(\lambda_3 r))\frac{1}{\rho_1 s^3} + C_{ij}\frac{1}{s}K_0(\lambda_3 r), \quad (47a)$$

$$\tilde{G}_{3i} = -\frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{\alpha_2}{\zeta}\frac{\lambda_1}{\lambda_1^2-\lambda_2^2}K_1(\lambda_1 r) + \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{\alpha_2}{\zeta}\frac{\lambda_2}{\lambda_1^2-\lambda_2^2}K_1(\lambda_2 r), \quad (47b)$$

$$\tilde{G}_{i3} = \frac{1}{s}\tilde{G}_{3i}, \quad (47c)$$

$$\tilde{G}_{33} = \frac{1}{2\pi\zeta s}\frac{1}{\lambda_1^2-\lambda_2^2}[(\lambda_1^2-\Lambda^2)K_0(\lambda_1 r) - (\lambda_2^2-\Lambda^2)K_0(\lambda_2 r)], \quad (47d)$$

where:

$$i,j = 1,2, \quad r^2 = x_i x_i$$
$$A_{ij} = \frac{1}{2\pi}\left(\frac{2x_i x_j}{r^3} - \frac{\delta_{ij}}{r}\right)$$
$$B_{ij} = \frac{1}{2\pi}\frac{x_i x_j}{r^2}$$
$$C_{ij} = \frac{\delta_{ij}}{2\pi\mu}, \quad (48)$$

in which $K_0(\lambda_i r)$ and $K_1(\lambda_i r)$ are the modified Bessel functions of the second kind of zero and first order. In the above Laplace transform domain solutions, i.e. eqns (47), $\tilde{G}_{ij}$ is the displacement of the solid skeleton in $i$-th direction due to the unit Heaviside line force in $j$-direction. Whereas $\tilde{G}_{3j}$ is the fluid pressure due to the unit Heaviside line force in $j$-th direction. Also $\tilde{G}_{i3}$ is the displacement of the solid skeleton in $i$-th direction due to the unit Heaviside rate of fluid line injection into fluid. And $\tilde{G}_{33}$ is the fluid pressure due to fluid injection.

In view of eqns (47) and (48), we see that for displacement generated by a line force there are two compressional waves, i.e. slow compressional wave (defined by Biot as $P_2$-wave), which is purely diffusive and is associated with $K_0(\lambda_1 r)$, $K_1(\lambda_1 r)$, fast compressional wave (defined by Biot as $P_1$-wave), which is equivalent to the elastodynamic pressure wave and associated with $K_0(\lambda_2 r)$, $K_1(\lambda_2 r)$ and the shear wave ($S$-wave) associated with $K_0(\lambda_3 r)$, $K_1(\lambda_3 r)$. The displacement is plane symmetrical around direction of line force. The contribution of $S$-wave in the pressure due to point force is obviously zero. Also there is no shear wave in the field radiated by a fluid line injection. This is because the shear wave can not significantly transmit through fluids. The displacement and pressure due to fluid injection presents a radial symmetry centred on fluid line injection. While pressure due to the line force $\tilde{G}_{3j}$ shows antisymmetry about the coordinate axis which is perpendicular to the direction of the line force (i.e. opposite sign of the values). In the corresponding frequency domain, $\lambda_1$ and $\lambda_2$ are the wave numbers of the slow compressional waves and fast compressional waves, while $\lambda_3$ is the wave number of the shear waves.

We may easily verify that each column-vector in the fundamental solution matrix $\tilde{\mathbf{G}}$ possesses a unique singularity at the point $x = 0$ of the order ln $r$. Obviously each column of the fundamental solution matrix $\tilde{\mathbf{G}}$ satisifes the system equation (21). Since the matrix $\tilde{\mathbf{G}}$ is unsymmetrical, as shown by eqns (47b) and (47c), its rows considered as vectors do not satisfy eqn (21).

## VERIFICATION OF THE SOLUTIONS

Of course, it is desirable to provide a means of checking the validity of the Green's function presented in the previous section. First, investigate the solution form as $\kappa$

approaches infinity and $\rho_f$, $m$ approach zero, to see if they would exactly take the same form as elastodynamic fundamental solution in Laplace transform domain. Next, the behaviour of the response function will be examined at very long times, as $t$ approaches infinity.

*Limiting case 1 : elastodynamics*

The Green's function, i.e. eqns (47) should reduce to the well-known elastodynamic line load solution by letting $\kappa$ approach infinity, $\rho_f$ and $m$ equal zero. Thus eqns (29a–g) yield:

$$\frac{1}{\zeta} = 0, \tag{49a}$$

$$\alpha_1 = \alpha_2 = \alpha, \tag{49b}$$

$$\rho_1 = \rho, \tag{49c}$$

$$\Lambda^2 = \frac{\rho}{\lambda+2\mu}s^2, \tag{49d}$$

$$\lambda_1^2 = \frac{\rho}{\lambda+2\mu}s^2, \tag{49e}$$

$$\lambda_2^2 = 0, \tag{49f}$$

$$\lambda_3^2 = \frac{\rho}{\mu}s^2. \tag{49g}$$

If we make the substitution

$$K_2(\lambda_i r) = K_0(\lambda_i r) + \frac{2}{\lambda_i r}K_1(\lambda_i r), \tag{50}$$

and combine eqns (49a–g) with eqns (47a–d), then the two-dimensional fundamental solutions reduce to:

$$\tilde{G}_{\alpha\beta}(x,\xi,s) = \frac{1}{2\pi\rho C_2^2}\left(a\delta_{\alpha\beta} - b\frac{x_\alpha x_\beta}{r^2}\right)\frac{1}{s}, \tag{51a}$$

$$\tilde{G}_{3\beta}(x,\xi,s) = 0, \tag{51b}$$

$$\tilde{G}_{\alpha 3}(x,\xi,s) = 0, \tag{51c}$$

$$\tilde{G}_{33}(x,\xi,s) = 0, \tag{51d}$$

where $\alpha, \beta = 1, 2$:

$$a = K_0\left(\frac{sr}{C_2}\right) + \frac{C_2}{sr}\left(K_1\left(\frac{sr}{C_2}\right) - \frac{C_2}{C_1}K_1\left(\frac{sr}{C_1}\right)\right), \tag{52a}$$

$$b = K_2\left(\frac{sr}{C_2}\right) - \frac{C_2^2}{C_1^2}K_2\left(\frac{sr}{C_1}\right), \tag{52b}$$

$$C_1^2 = \frac{\lambda+2\mu}{\rho}, \tag{52c}$$

$$C_2^2 = \frac{\mu}{\rho}. \tag{52d}$$

After being multiplied by Laplace parameter $s$ (since the present $\tilde{G}_{\alpha\beta}$ is due to a line force which is a step function $H(t)$ in time), eqn (51a) reduces to the corresponding elastodynamics Green's function due to a line force with Dirac delta function in time (Doyle, 1966; Cruse and Rizzo, 1968a, b). This supports the fact that the Laplace transform domain fundamental solutions derived in previous sections are likely to be correct.

*Limiting case 2: steady state poroelasticity*

Now we check transform domain fundamental solution, by letting $t \to \infty$, to see if it takes exactly the same form as steady-state poroelastic fundamental solution. We are to investigate this limiting case by means of letting $r \to 0$. The reason this approach can be used is because the transform domain fundamental solutions exhibit "static-like" behaviour as their argument $\lambda_i r \to 0$.

First, consider $\tilde{G}_{33}(x, s)$, the pressure due to the unit step line fluid injection. As $r \to 0$, so does the argument of the modified Bessel functions. The limiting form of $K_0(z)$ as $z \to 0$ is:

$$K_0(z) = -\ln z. \tag{53}$$

Thus the eqn (47d) reduces to:

$$s\tilde{G}_{33}(\mathbf{x}, s) = \frac{1}{2\pi\kappa}\ln\frac{1}{r} + s\frac{m}{2\pi}\ln\frac{1}{r} + \frac{1}{2\pi\zeta}((\Lambda^2-\lambda_1^2)\ln\lambda_1 - (\Lambda^2-\lambda_2^2)\ln\lambda_2)\frac{1}{\lambda_1^2-\lambda_2^2}. \tag{54}$$

The first term is the fundamental solution to the two-dimensional Laplace equation, and the second term contributes nothing to the steady-state, while the third term is a nonsingular constant which adds nothing to the solution.

Second, consider $G_{i3}$, eqn (47c) can be rewritten as:

$$\begin{aligned} s\tilde{G}_{i3}(\mathbf{x}, s) = &-\frac{1}{4\pi(\lambda+2\mu)}\frac{1}{\lambda_1^2-\lambda_2^2}\left(\frac{\alpha}{\kappa} + s(\alpha m - \rho_f)\right)x_i f(r) \\ &-\frac{1}{4\pi(\lambda+2\mu)}\frac{1}{\lambda_1^2-\lambda_2^2}\left(\frac{\alpha}{\kappa} + s(\alpha m - \rho_f)\right)x_i(-\lambda_1^2 K_0(\lambda_1 r) + \lambda_2^2 K_0(\lambda_2 r)), \end{aligned} \tag{55}$$

where

$$f(r) = \frac{\lambda_1^2 r^2 K_2(\lambda_1 r) - \lambda_2^2 r^2 K_2(\lambda_2 r)}{r^2}. \tag{56}$$

It is therefore of interest to examine the value of $f(r)$ when $r \to 0$. Accordingly, we note that $K_2(z) = 2/z^2$ as $z \to 0$. Thus the numerator and denominator of eqn (56) tend to zero as $r \to 0$. This gives us the case 0/0. We apply L'Hôpital's rule and note that $K_1(z) = 1/z$ as $z \to 0$. Consequently:

$$\begin{aligned} \lim_{r\to 0} f(r) &= -\tfrac{1}{2}r\lambda_1^3 K_1(\lambda_1 r) + \tfrac{1}{2}r\lambda_2^3 K_1(\lambda_2 r) \\ &= -\tfrac{1}{2}(\lambda_1^2 - \lambda_2^2). \end{aligned} \tag{57}$$

Substituting eqn (57) into eqn (55) and using (53), we find ultimately:

$$\begin{aligned} s\tilde{G}_{i3}(\mathbf{x}, s) = &\frac{r}{4\pi}\frac{\alpha}{\kappa(\lambda+2\mu)}\left(\frac{x_i}{r}(\tfrac{1}{2} - \ln r)\right) + \frac{r}{4\pi}\frac{\alpha m - \rho_f}{(\lambda+2\mu)}\left(\frac{x_i}{r}(\tfrac{1}{2} - \ln r)\right)s \\ &-\frac{1}{4\pi(\lambda+2\mu)}\left(\frac{\alpha}{\kappa} + s(\alpha m - \rho_f)\right)\frac{\lambda_1^2\ln\lambda_1 - \lambda_2^2\ln\lambda_2}{\lambda_1^2-\lambda_2^2}x_i \quad \text{for} \quad r \to 0, \end{aligned} \tag{58}$$

where the first term represents the steady-state displacement field due to a constant unit line fluid injection, and the second term contributes nothing to the steady-state, while the third term is a linear function of $x_i$, with constant coefficient, which adds nothing to the solution.

Next consider $\tilde{G}_{3i}$. Obviously from eqn (47b), we have:

$$s\tilde{G}_{3i}(\mathbf{x}, s) = 0 + s^2 \tilde{G}_{i3}(\mathbf{x}, \xi, s). \tag{59}$$

The first term is zero, which means that pressure due to constant unit line force in the $i$-direction at very long times vanishes, while the second term contributes nothing to the steady-state.

Finally we discuss $G_{ij}$, which can be more compactly expressed as:

$$s\tilde{G}_{ij}(\mathbf{x}, s) = \frac{x_i x_j}{2\pi r^2} \frac{1}{\rho_1 s^2} h(r) - \frac{\delta_{ij}}{4\pi\rho_1 s^2} h(r) - \frac{\delta_{ij}}{4\pi\rho_1 s^2} \left\{ - \frac{\lambda_1^2(\Lambda^2 - \lambda_2^2)}{\lambda_1^2 - \lambda_2^2} K_0(\lambda_1 r) + \frac{\lambda_2^2(\Lambda^2 - \lambda_1^2)}{\lambda_1^2 - \lambda_2^2} K_0(\lambda_2 r) + \lambda_3^2 K_0(\lambda_3 r) \right\} + \frac{\delta_{ij}}{2\pi\mu} K_0(\lambda_3 r), \tag{60}$$

where

$$h(r) = \frac{1}{r^2} \left\{ \frac{\lambda_1^2(\Lambda^2 - \lambda_2^2)}{\lambda_1^2 - \lambda_2^2} r^2 K_2(\lambda_1 r) - \frac{\lambda_2^2(\Lambda^2 - \lambda_1^2)}{\lambda_1^2 - \lambda_2^2} r^2 K_2(\lambda_2 r) - \lambda_3^2 r^2 K_2(\lambda_3 r) \right\}. \tag{61}$$

In order to determine the value of $h(r)$ when $r \to 0$, we take into account that $K_2(z) = 2/z^2$ as $z \to 0$, this is the case 0/0. By applying L'Hôpital's rule to eqn (61), one obtains:

$$h(r) = \tfrac{1}{2}(\lambda_3^2 - \Lambda^2), \tag{62}$$

for $r \to 0$. Combine eqns (60) and (62), and also keep in mind $K_0(z) = -\ln z$ as $z \to 0$, after some algebraic manipulation, we arrive at:

$$s\tilde{G}_{ij}(\mathbf{x}, s) = \frac{1}{8\pi} \frac{1}{\mu(1-\nu)} \left\{ \frac{x_i x_j}{r^2} - \delta_{ij}(3 - 4\nu) \ln r \right\} + \delta_{ij} \left\{ - \frac{1}{4\pi\rho_1 s^2} \frac{\lambda_1^2(\Lambda^2 - \lambda_2^2)}{\lambda_1^2 - \lambda_2^2} \ln \lambda_1 + \frac{1}{4\pi\rho_1 s^2} \frac{\lambda_2^2(\Lambda^2 - \lambda_1^2)}{\lambda_1^2 - \lambda_2^2} \ln \lambda_2 - \frac{1}{4\pi\mu} \ln \lambda_3 + \frac{1}{8\pi} \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) \right\}, \tag{63}$$

where the first term is the fundamental solution to the two-dimensional elastostatics, while the second term is a nonsingular constant which adds nothing to the solution. Checking this steady state poroelastic limiting case again confirms the correctness of the two-dimensional Laplace transform domain fundamental solutions.

## TRANSIENT FUNDAMENTAL SOLUTION (LIMITING CASE-WAVEFRONT FORMULAS)

With the Laplace transform fundamental solution being derived and verified, we now proceed to derive its counterpart in the time domain by using analytical inversion. Due to the complexity of Laplace domain Green's functions for two-dimensions, the analytical solutions have long been thought to be extremely difficult, or even considered impossible. As a result, very little progress has been made since Biot's equations of dynamic poroelasticity were published in the 1950s.

Ghaboussi and Wilson (1972), while presenting a finite element solution to an infinite one dimensional problem, noted that an analytical solution could not be obtained for this problem. Norris (1985) emphasized that the presence of viscous damping ($1/\kappa$) in the system equations makes the inverse transform impossible to do in closed form. Garg *et al.* (1974) solved an infinite long linear elastic fluid-saturated soil column subjected to a Heaviside step function velocity boundary condition at one end for the limiting cases of small viscous coupling and strong viscous coupling by using asymptotic expansions. He also noted that it is not possible to obtain close-form solution of this problem for the general case. Simon *et al.* (1984) derived an analytical solution for the same one-dimensional problem assuming the solid and fluid phases to be "dynamically compatible" (when a dynamic compatibility relation holds, the porous medium can support a non-dissipative wave in which the fluid moves with the solid).

Here we show that time domain fundamental solution of Biot's full dynamic poroelasticity for the two-dimensional case is possible not only for the limiting cases but also for the general case even with conditions of dynamic compatibility not being satisfied. Through the current paper and Part II (Chen, 1994) we should always bear in mind that for dimensional form eqns (8) and (9) will be followed, while for nondimensional form eqns (15) and (16) will be followed. For convenience all the superscript * for nondimensional material parameters have been dropped. The two roots of eqns (29a) and (29b) are $\lambda_1^2$ and $\lambda_2^2$ and are presented here as:

$$\lambda_{1,2}^2 = \frac{1}{2}\left\{\frac{s}{\zeta}\left(\frac{1}{Q}+\frac{\alpha_1^2}{\lambda+2\mu}\right)+\Lambda^2 \pm \sqrt{\left[\frac{s}{\zeta}\left(\frac{1}{Q}+\frac{\alpha_1^2}{\lambda+2\mu}\right)+\Lambda^2\right]^2 - 4\frac{s}{\zeta}\frac{\Lambda^2}{Q}}\right\}. \tag{64}$$

From eqn (64) we directly have:

$$\lambda_1^2-\lambda_2^2 = \left\{\left[\Lambda^2+\frac{s}{\zeta}\left(\frac{1}{Q}+\frac{\alpha_1^2}{\lambda+2\mu}\right)\right]^2 - 4\frac{s}{\zeta}\frac{1}{Q}\Lambda^2\right\}^{1/2}. \tag{65}$$

Now considering that the solution is assumed to be finite at $r_i \to \infty$ or $x_i \to \infty$ and no reflections are possible, eqn (64) can be more concisely expressed as:

$$\lambda_{1,2} = \left\{\frac{1}{2}\left[\frac{s}{\zeta}\left(\frac{1}{Q}+\frac{\alpha_1^2}{\lambda+2\mu}\right)+\Lambda^2 \pm (\lambda_1^2-\lambda_2^2)\right]\right\}^{1/2}. \tag{66}$$

According to the theory of the Laplace transform, the functional behaviour of some image function $F(s)$ as $s \to \infty$ is determined by the functional behaviour of the corresponding original function $f(t)$ near $t = 0$; one could say that the function behaviour of $f(t)$ near $t = 0$ is mapped onto the functional behaviour of the $L$-transform $F(s)$ near $s \to \infty$. The Tauberian theorem for hyperbolic equations predicts the behaviour, as $s \to \infty$, of the transform domain fundamental solutions, gives the jump at wavefronts and also the approximate nature of the disturbance near the arrival time.

Hence substituting eqns (29d,e,f) into eqn (64), for small values of $1/s$ (corresponding to small times), eqn (65) yields, to a second order of approximation in $1/s$,

$$\lambda_1^2-\lambda_2^2 \approx \sqrt{a_3}\left[s^2+a_1\frac{s}{\kappa}+a_2\frac{1}{\kappa^2}\right], \tag{67}$$

where

$$a_1 = \frac{a_4}{2a_3}$$

$$a_2 = \frac{1}{2}\left(-\frac{1}{4}\frac{a_4^2}{a_3^2}+\frac{a_5}{a_3}\right)$$

$$a_3 = \left(\frac{\rho+\alpha^2 m-2\alpha\rho_f}{\lambda+2\mu}+\frac{m}{Q}\right)^2+\frac{4(\rho_f^2-\rho m)}{Q(\lambda+2\mu)}$$

$$a_4 = 2\left[\frac{-\rho+2\alpha^2 m-2\alpha\rho_f}{Q(\lambda+2\mu)}+\frac{\alpha^2(\rho+\alpha^2 m-2\alpha\rho_f)}{(\lambda+2\mu)^2}+\frac{m}{Q^2}\right]$$

$$a_5 = \left(\frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}\right)^2. \tag{68}$$

For convenience of later derivation eqn (67) can be given alternatively as:

$$\lambda_1^2-\lambda_2^2 \approx \sqrt{a_3}(s+b)(s+c), \tag{69}$$

where $b$, $c$ are the roots of quadratic polynomials eqn (69):

$$b,c = -\frac{1}{2}\left\{-\frac{a_1}{\kappa}\pm\sqrt{\left(\frac{a_1}{\kappa}\right)^2-\frac{4a_2}{\kappa^2}}\right\}. \tag{70}$$

Substituting eqns (67) and (29d–f) into (66) and with some algebraic manipulation, one obtains:

$$\lambda_{1,2} \approx \frac{1}{c_{d,p}}\sqrt{(s+\eta_{d,p})^2-\xi_{d,p}^2}, \tag{71}$$

where

$$c_{d,p} = \frac{\sqrt{2}}{\sqrt{a_6\pm\sqrt{a_3}}}$$

$$\eta_{d,p} = \frac{1}{2}\frac{a_7\pm\dfrac{a_4}{2\sqrt{a_3}}}{a_6\pm\sqrt{a_3}}\frac{1}{\kappa}$$

$$\xi_{d,p} = \left\{(\eta_{d,p})^2\mp\frac{\dfrac{1}{2}\left(-\dfrac{a_4^2}{4a_3^{3/2}}+\dfrac{a_5}{\sqrt{a_3}}\right)\dfrac{1}{\kappa^2}}{a_6\pm\sqrt{a_3}}\right\}^{1/2}$$

$$a_6 = \frac{\rho+\alpha^2 m-2\alpha\rho_f}{\lambda+2\mu}+\frac{m}{Q}$$

$$a_7 = \frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}. \tag{72}$$

In the above, $c_d$ and $c_p$ are the velocities of diffusive wave ($P_2$-wave) and pressure wave ($P_1$-wave) respectively, $\eta_d$ and $\eta_p$ are the dissipation factors, which reflect the damping effects of the diffusive wave and pressure wave, respectively.

At this stage it is helpful to find out the range of validity for the above approximation when the solutions are in dimensional form.

Following the same method, we can expand eqn (65) in powers of $1/\kappa$ to a second order of approximation for large permeability coefficient $\kappa$. If $\kappa$ is large, then $1/\kappa$ is small (corresponding to small viscous coupling) and the resulting expression will be convergent. Fortunately this approximation takes exactly the same form as given in eqn (67). Carefully scrutinizing eqn (67) reveals that after taking common factor $s^2$ outside the parenthesis of eqn (67), the expression remaining within the parenthesis is a quadratic polynomial in $1/ks$. Thus the asymptotic expansion of eqn (65) in powers of $1/\kappa$, $1/s$ or $1/\kappa s$, respectively leads to the same result, i.e. eqn (67). Now we come to the conclusion that as long as $\kappa s$ is large, the asymptotic expansion is justified, if $s$ or $\kappa$ are large the time domain solution will be valid for small $t$. It is interesting to note that the increase in $\kappa$ always means an increase in

$t$, during which the time domain solution is valid. Referring to the equality $\kappa = k/\eta$ we see that this could occur if, for example, the fluid viscosity $\eta$ is small or the intrinsic permeability $k$ large. The former is the case for superfluid (Plona, 1980) *He* in the pores, the so called "superleak" (Johnson, 1982) while large intrinsic permeability is obtained in experimental situations in which the fluid can percolate easily, for example, glass beads immersed in fluid.

Utilizing the similar procedure and expanding eqn (29c) to a second order of approximation in $1/ks$ for large value of $ks$, one obtains:

$$\lambda_3 = \sqrt{\frac{\rho-\rho_f^2/m}{\mu}}\,s\left(1+\frac{\rho_f^2/m^2}{\rho-\rho_f^2/m}\frac{1}{\kappa s}-\frac{\rho_f^2/m^3}{\rho-\rho_f^2/m}\frac{1}{k^2s^2}\right)^{1/2}. \tag{73}$$

Equation (73) can now be rewritten as:

$$\lambda_3 = \frac{1}{c_s}\sqrt{(s+\eta_s)^2-\xi_s^2}, \tag{74}$$

where

$$c_s = \sqrt{\frac{\mu}{\rho}}\frac{1}{\sqrt{1-\dfrac{\rho_f^2}{\rho m}}}$$

$$\eta_s = \frac{1}{2\kappa m}\frac{\rho_f^2}{m\rho-\rho_f^2}$$

$$\xi_s = \left(\eta_s^2+\frac{\rho_f^2}{(m\rho-\rho_f^2)m^2\kappa^2}\right)^{1/2}. \tag{75}$$

In the above $c_s$ and $\eta_s$ are the velociity and dissipation factor of the shear wave (or equivoluminal wave), respectively.

With the asymptotic expansion derived for $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_1^2-\lambda_2^2$, we now proceed to solve for the two-dimensional Green's function for short time using the Laplace transform. Compared with the three-dimensional dynamic transient poroelastic fundamental solutions (Chen, 1994), the corresponding two-dimensional ones are even more difficult to obtain. This complexity is due to the existence of modified Bessel functions (zero order and first order) in the transform domain solutions, while in three-dimensions there are only exponential functions which are obviously easier to handle.

*Function $G_{ij}$*

The eqn (47a), as it stands, is too complicated for direct inversion. However, inspection of it reveals that it can be rearranged as:

$$\begin{aligned}\tilde{G}_{ij} = &\left(-B_{ij}e_1\frac{1}{\lambda_1^2-\lambda_2^2}-B_{ij}e_2\frac{s}{\lambda_1^2-\lambda_2^2}-B_{ij}e_{31}\frac{1}{s}\right)K_0(\lambda_1 r)\\ &+\left(-A_{ij}e_1\frac{1}{\lambda_1^2-\lambda_2^2}-A_{ij}e_2\frac{s}{\lambda_1^2-\lambda_2^2}-A_{ij}e_{31}\frac{1}{s}\right)\frac{1}{\lambda_1}K_1(\lambda_1 r)\\ &+\left(B_{ij}e_1\frac{1}{\lambda_1^2-\lambda_2^2}+B_{ij}e_2\frac{s}{\lambda_1^2-\lambda_2^2}+B_{ij}e_{32}\frac{1}{s}\right)K_0(\lambda_2 r)\\ &+\left(A_{ij}e_1\frac{1}{\lambda_1^2-\lambda_2^2}+A_{ij}e_2\frac{s}{\lambda_1^2-\lambda_2^2}+A_{ij}e_{32}\frac{1}{s}\right)\frac{1}{\lambda_2}K_1(\lambda_2 r)\\ &+\left(-B_{ij}\frac{1}{\mu}\frac{1}{s}+C_{ij}\frac{1}{s}\right)K_0(\lambda_3 r)+\left(-A_{ij}\frac{1}{\mu}\frac{1}{s}\right)\frac{1}{\lambda_3}K_1(\lambda_3 r),\end{aligned} \tag{76}$$

where

$$e_1 = \frac{1}{2(\lambda+2\mu)\kappa}\left(\frac{1}{Q} - \frac{\alpha^2}{\lambda+2\mu}\right)$$

$$e_2 = \frac{1}{2(\lambda+2\mu)}\left(\frac{m}{Q} + \frac{-\alpha^2 m - \rho + 2\alpha\rho_f}{\lambda+2\mu}\right)$$

$$e_{31} = -\frac{1}{2(\lambda+2\mu)}$$

$$e_{32} = \frac{1}{2(\lambda+2\mu)}. \tag{77}$$

Using eqn (76) in conjunction with eqn (69), we finally have:

$$\tilde{G}_{ij} = \left\{f_1\frac{1}{s} + f_2\frac{1}{s+c} + f_3\frac{1}{(s+b)(s+c)}\right\}K_0(\lambda_1 r) + \left\{f_4\frac{1}{s} + f_5\frac{1}{s+c} + f_6\frac{1}{(s+b)(s+c)}\right\}$$

$$\times \frac{1}{\lambda_1}K_1(\lambda_1 r) + \left\{g_1\frac{1}{s} + g_2\frac{1}{s+c} + g_3\frac{1}{(s+b)(s+c)}\right\}K_0(\lambda_2 r)$$

$$+ \left\{g_4\frac{1}{s} + g_5\frac{1}{s+c} + g_6\frac{1}{(s+b)(s+c)}\right\}\frac{1}{\lambda_2}K_1(\lambda_2 r) + h_1\frac{1}{s}K_0(\lambda_3 r) + h_2\frac{1}{s}\frac{1}{\lambda_3}K_1(\lambda_3 r), \tag{78}$$

where

$$f_1 = -B_{ij}e_{31} \qquad g_1 = B_{ij}e_{32} \qquad h_1 = -\frac{B_{ij}}{\mu} + C_{ij}$$

$$f_2 = -\frac{B_{ij}e_2}{\sqrt{a_3}} \qquad g_2 = \frac{B_{ij}e_2}{\sqrt{a_3}} \qquad h_2 = -\frac{A_{ij}}{\mu}$$

$$f_3 = \frac{-B_{ij}e_1 + B_{ij}e_2 b}{\sqrt{a_3}} \qquad g_3 = \frac{B_{ij}e_1 - B_{ij}e_2 b}{\sqrt{a_3}}$$

$$f_4 = -A_{ij}e_{31} \qquad g_4 = A_{ij}e_{32}$$

$$f_5 = -\frac{A_{ij}e_2}{\sqrt{a_3}} \qquad g_5 = \frac{A_{ij}e_2}{\sqrt{a_3}}$$

$$f_6 = \frac{-A_{ij}e_1 + A_{ij}e_2 b}{\sqrt{a_3}}. \qquad g_6 = \frac{A_{ij}e_1 - A_{ij}e_2 b}{\sqrt{a_3}}. \tag{79}$$

To invert eqn (78), we first observe the following formulas (Abramowitz and Stegun, 1965; Roberts and Kaufman, 1966)):

$$L^{-1}\{K_0[a(s^2-b^2)^{1/2}]\} = (t^2-a^2)^{-1/2}\cosh[b(t^2-a^2)^{1/2}]$$

$$L^{-1}\{(s^2-b^2)^{-1/2}K_1[a(s^2-b^2)^{1/2}]\} = (ab)^{-1}\sinh[b(t^2-a^2)^{1/2}]. \tag{80}$$

Using eqn (80) in conjunction with eqn (78) and employing properties of the inverse transforms, we finally obtain:

$$G_{ij} = \int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12}e^{-c(t-\tau)} + P_{13}) \frac{e^{-\eta_d\tau}}{\sqrt{\tau^2 - r^2/c_d^2}}$$

$$\times \cosh\left(\xi_d\sqrt{\tau^2 - r^2/c_d^2}\right) \mathrm{d}\tau\, H(t - r/c_d) + \int_{r/c_d}^{t} (P_{21}e^{-b(t-\tau)}$$

$$+ P_{22}e^{-c(t-\tau)} + P_{23})e^{-\eta_d\tau} \sinh\left(\xi_d\sqrt{\tau^2 - r^2/c_d^2}\right) \mathrm{d}\tau\, H(t - r/c_d) + \int_{r/c_p}^{t} (P_{31}e^{-b(t-\tau)}$$

$$+ P_{32}e^{-c(t-\tau)} + P_{33}) \frac{e^{-\eta_p\tau}}{\sqrt{\tau^2 - r^2/c_p^2}} \cosh\left(\xi_p\sqrt{\tau^2 - r^2/c_p^2}\right) \mathrm{d}\tau\, H(t - r/c_p)$$

$$+ \int_{r/c_p}^{t} (P_{41}e^{-b(t-\tau)} + P_{42}e^{-c(t-\tau)} + P_{43})e^{-\eta_p\tau} \sinh\left(\xi_p\sqrt{\tau^2 - r^2/c_p^2}\right) \mathrm{d}\tau\, H(t - r/c_p)$$

$$+ \int_{r/c_s}^{t} P_{51} \frac{e^{-\eta_s\tau}}{\sqrt{\tau^2 - r^2/c_s^2}} \cosh\left(\xi_s\sqrt{\tau^2 - r^2/c_s^2}\right) \mathrm{d}\tau\, H(t - r/c_s)$$

$$+ \int_{r/c_s}^{t} P_{61}e^{-\eta_s\tau} \sinh\left(\xi_s\sqrt{\tau^2 - r^2/c_s^2}\right) \mathrm{d}\tau\, H(t - r/c_s), \tag{81}$$

where

$$\begin{aligned}
P_{11} &= \frac{f_3}{c-b} & P_{31} &= \frac{g_3}{c-b} \\
P_{12} &= f_2 - \frac{f_3}{c-b} & P_{32} &= g_2 - \frac{g_3}{c-b} \\
P_{13} &= f_1 & P_{33} &= g_1 \\
P_{21} &= \frac{f_6}{c-b}\frac{c_d^2}{r\xi_d} & P_{41} &= \frac{g_6}{c-b}\frac{c_p^2}{r\xi_p} \\
P_{22} &= \left(f_5 - \frac{f_6}{c-b}\right)\frac{c_d^2}{r\xi_d} & P_{42} &= \left(g_5 - \frac{g_6}{c-b}\right)\frac{c_p^2}{r\xi_p} \\
P_{23} &= f_4\frac{c_d^2}{r\xi_d} & P_{43} &= g_4\frac{c_p^2}{r\xi_p} \\
P_{51} &= h_1 & P_{61} &= h_2\frac{c_s^2}{r\xi_s}.
\end{aligned} \tag{82}$$

Let us observe that in eqn (81) we have waves of three types: the first corresponding to the highly dissipative wave traveling with speed $c_d$ and dissipation factor $\eta_d$, the second corresponding to the pure wave traveling with speed $c_p$ and dissipation factor $\eta_p$ and the third corresponding to the shear wave traveling with speed $c_s$ and dissipation factor $\eta_s$.

*Function $G_{3i}$*

$G_{3i}$ is that given by eqn (47b) and can be put in a more convenient form:

$$\tilde{G}_{3i} = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left\{-e_1 s - e_2 - e_3\frac{s}{\lambda_1^2-\lambda_2^2} - e_4\frac{s^2}{\lambda_1^2-\lambda_2^2} - e_5\frac{s^3}{\lambda_1^2-\lambda_2^2}\right\}\frac{1}{\lambda_1}K_1(\lambda_1 r)$$

$$+ \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left\{-e_1 s - e_2 + e_3\frac{s}{\lambda_1^2-\lambda_2^2} + e_4\frac{s^2}{\lambda_1^2-\lambda_2^2} + e_5\frac{s^3}{\lambda_1^2-\lambda_2^2}\right\}\frac{1}{\lambda_2}K_1(\lambda_2 r), \tag{83}$$

where

$$e_1 = \tfrac{1}{2}(\alpha m - \rho_f)$$
$$e_2 = \frac{\alpha}{2\kappa}$$
$$e_3 = \frac{1}{2}\left(\frac{1}{Q} + \frac{\alpha^2}{\lambda+2\mu}\right)\frac{\alpha}{\kappa^2}$$
$$e_4 = \frac{1}{2}\left[\frac{2\alpha m - \rho_f}{Q} + \frac{2\alpha^3 m - 3\alpha^2\rho_f + \alpha\rho}{\lambda+2\mu}\right]\frac{1}{\kappa}$$
$$e_5 = \frac{1}{2}\left(\frac{m}{Q} + \frac{\alpha^2 m - 2\alpha\rho_f + \rho}{\lambda+2\mu}\right)(\alpha m - \rho_f). \tag{84}$$

Substitution of eqn (69) into eqn (83) yields:

$$\tilde{G}_{3i} = \left\{f_1 s + f_2 + f_3\frac{1}{s+c} + f_4\frac{1}{(s+b)(s+c)}\right\}\frac{1}{\lambda_1}K_1(\lambda_1 r)$$
$$+\left\{g_1 s + g_2 + g_3\frac{1}{s+c} + g_4\frac{1}{(s+b)(s+c)}\right\}\frac{1}{\lambda_2}K_1(\lambda_2 r), \tag{85}$$

where

$$f_1 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_1 - \frac{e_5}{\sqrt{a_3}}\right)$$
$$f_2 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_2 - \frac{e_4}{\sqrt{a_3}} + \frac{e_5(b+c)}{\sqrt{a_3}}\right)$$
$$f_3 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{-e_3 + e_4(b+c) - e_5(b^2+bc+c^2)}{\sqrt{a_3}}$$
$$f_4 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{e_3 b - e_4 b^2 + e_5 b^3}{\sqrt{a_3}}$$
$$g_1 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_1 + \frac{e_5}{\sqrt{a_3}}\right)$$
$$g_2 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_2 + \frac{e_4}{\sqrt{a_3}} - \frac{e_5(b+c)}{\sqrt{a_3}}\right)$$
$$g_3 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{e_3 - e_4(b+c) + e_5(b^2+bc+c^2)}{\sqrt{a_3}}$$
$$g_4 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{-e_3 b + e_4 b^2 - e_5 b^3}{\sqrt{a_3}}. \tag{86}$$

Since eqn (85) is expressed in terms of known transforms, we can invert it directly to give the following closed form solutions:

$$G_{3i} = \int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12}e^{-c(t-\tau)})e^{-\eta_d\tau} \sinh(\xi_d\sqrt{\tau^2 - r^2/c_d^2})\,\mathrm{d}\tau\, H(t - r/c_d)$$
$$+ P_{13}e^{-\eta_d t} \sinh(\xi_d\sqrt{t^2 - r^2/c_d^2})H(t - r/c_d) + P_{14}e^{-\eta_d t} \cosh(\xi_d\sqrt{t^2 - r^2/c_d^2})$$
$$\times \frac{t}{\sqrt{t^2 - r^2/c_d^2}} H(t - r/c_d) + P_{15}e^{-\eta_d t} \sinh(\xi_d\sqrt{t^2 - r^2/c_d^2})H(t - r/c_d)$$
$$+ \int_{r/c_p}^{t} (P_{21}e^{-b(t-\tau)} + P_{22}e^{-c(t-\tau)})e^{-\eta_p\tau} \sinh(\xi_p\sqrt{\tau^2 - r^2/c_p^2})\,\mathrm{d}\tau\, H(t - r/c_p)$$
$$+ P_{23}e^{-\eta_p t} \sinh(\xi_p\sqrt{t^2 - r^2/c_p^2})H(t - r/c_p) + P_{24}e^{-\eta_p t} \cosh(\xi_p\sqrt{t^2 - r^2/c_p^2})$$
$$\times \frac{t}{\sqrt{t^2 - r^2/c_p^2}} H(t - r/c_p) + P_{25}e^{-\eta_p t} \sinh(\xi_p\sqrt{t^2 - r^2/c_p^2})H(t - r/c_p), \quad (87)$$

where

$$P_{11} = \frac{f_4}{c-b}\frac{c_d^2}{r\xi_d} \qquad P_{21} = \frac{g_4}{c-b}\frac{c_p^2}{r\xi_p}$$
$$P_{12} = \left(f_3 - \frac{f_4}{c-b}\right)\frac{c_d^2}{r\xi_d} \qquad P_{22} = \left(g_3 - \frac{g_4}{c-b}\right)\frac{c_p^2}{r\xi_p}$$
$$P_{13} = -f_1\frac{c_d^2\eta_d}{r\xi_d} \qquad P_{23} = -g_1\frac{c_p^2\eta_p}{r\xi_p}$$
$$P_{14} = f_1\frac{c_d^2}{r} \qquad P_{24} = g_1\frac{c_p^2}{r}$$
$$P_{15} = f_2\frac{c_d^2}{r\xi_d} \qquad P_{25} = g_2\frac{c_p^2}{r\xi_p}. \quad (88)$$

Equation (87) has two parts: one damped disturbance with large dissipative factor $\eta_d$ propagating at speed $c_d$ and an almost undamped disturbance with very small dissipative factor $\eta_p$ traveling at speed $c_p$.

*Function $G_{i3}$*

By using the following relationship:

$$\tilde{G}_{i3} = \frac{1}{s}\tilde{G}_{3i} \quad (89)$$

along with eqn (85), we immediately have:

$$\tilde{G}_{i3} = \left\{f_1 + f_2\frac{1}{s} + f_3\frac{1}{s(s+c)} + f_4\frac{1}{s(s+b)(s+c)}\right\}\frac{1}{\lambda_1}K_1(\lambda_1 r)$$
$$+ \left\{g_1 + g_2\frac{1}{s} + g_3\frac{1}{s(s+c)} + g_4\frac{1}{s(s+b)(s+c)}\right\}\frac{1}{\lambda_2}K_1(\lambda_2 r), \quad (90)$$

where $f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4$, have the same value as in eqn (86). After applying Laplace inversion, the solution takes the form:

$$G_{i3} = \int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12}e^{-c(t-\tau)} + P_{13})e^{-\eta_d\tau} \sinh(\xi_d\sqrt{\tau^2 - r^2/c_d^2})\,\mathrm{d}\tau\, H(t - r/c_d)$$
$$+ P_{14}e^{-\eta_d t} \sinh(\xi_d\sqrt{t^2 - r^2/c_d^2})H(t - r/c_d) + \int_{r/c_p}^{t} (P_{21}e^{-b(t-\tau)}$$

$$+P_{22}e^{-c(t-\tau)}+P_{23})e^{-\eta_p\tau}\sinh\left(\xi_p\sqrt{\tau^2-r^2/c_p^2}\right)\mathrm{d}\tau\, H(t-r/c_p)$$
$$+P_{24}e^{-\eta_p t}\sinh\left(\xi_p\sqrt{t^2-r^2/c_p^2}\right)H(t-r/c_p) \tag{91}$$

where

$$P_{11}=\frac{f_4}{b(b-c)}\frac{c_d^2}{r\xi_d}$$
$$P_{12}=\left(-\frac{f_3}{c}-\frac{f_4}{c(b-c)}\right)\frac{c_d^2}{r\xi_d}$$
$$P_{13}=\left(f_2+\frac{f_3}{c}+\frac{f_4}{bc}\right)\frac{c_d^2}{r\xi_d}$$
$$P_{14}=f_1\frac{c_d^2}{r\xi_d}$$
$$P_{21}=\frac{g_4}{b(b-c)}\frac{c_p^2}{r\xi_p}$$
$$P_{22}=\left(-\frac{g_3}{c}-\frac{g_4}{c(b-c)}\right)\frac{c_p^2}{r\xi_p}$$
$$P_{23}=\left(g_2+\frac{g_3}{c}+\frac{g_4}{bc}\right)\frac{c_p^2}{r\xi_p}$$
$$P_{24}=g_1\frac{c_p^2}{r\xi_p}. \tag{92}$$

It appears from eqn (91) that only two compressional waves radiate from a line source.

*Function $G_{33}$*

Considering eqn (47d) is not readily invertible, for the convenience of inversion, we rearrange it as:

$$\tilde{G}_{33}=\frac{1}{2\pi}\left\{e_1+e_2\frac{1}{s}+\frac{1}{\lambda_1^2-\lambda_2^2}(e_3+e_4s+e_5s^2)\right\}K_0(\lambda_1 r)$$
$$+\frac{1}{2\pi}\left\{e_1+e_2\frac{1}{s}-\frac{1}{\lambda_1^2-\lambda_2^2}(e_3+e_4s+e_5s^2)\right\}K_0(\lambda_2 r), \tag{93}$$

where:

$$e_1=\frac{m}{2}$$
$$e_2=\frac{1}{2\kappa}$$
$$e_3=\frac{1}{2\kappa^2}\left(\frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}\right)$$
$$e_4=\frac{m}{\kappa}\left(\frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}\right)-\frac{\rho+2\alpha\rho_f}{2\kappa(\lambda+2\mu)}$$
$$e_5=\frac{m^2}{2}\left(\frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}\right)-\frac{m(\rho+2\alpha\rho_f)}{2(\lambda+2\mu)}+\frac{\rho_f^2}{\lambda+2\mu}. \tag{94}$$

By introducing eqn (67) into eqn (93), we obtain:

$$\tilde{G}_{33} = \left\{f_1 + f_2\frac{1}{s} + f_3\frac{1}{s+c} + f_4\frac{1}{(s+b)(s+c)}\right\}K_0(\lambda_1 r) + \left\{g_1 + g_2\frac{1}{s} + g_3\frac{1}{s+c} + g_4\frac{1}{(s+b)(s+c)}\right\}K_0(\lambda_2 r), \quad (95)$$

where

$$\begin{aligned}
f_1 &= \frac{1}{2\pi}\left(e_1 + \frac{e_5}{\sqrt{a_3}}\right)\\
f_2 &= \frac{1}{2\pi}e_2\\
f_3 &= \frac{1}{2\pi}\frac{e_4 - e_5(b+c)}{\sqrt{a_3}}\\
f_4 &= \frac{1}{2\pi}\frac{e_3 - e_4 b + e_5 b^2}{\sqrt{a_3}}\\
g_1 &= \frac{1}{2\pi}\left(e_1 - \frac{e_5}{\sqrt{a_3}}\right)\\
g_2 &= \frac{1}{2\pi}e_2\\
g_3 &= \frac{1}{2\pi}\frac{-e_4 + e_5(b+c)}{\sqrt{a_3}}\\
g_4 &= \frac{1}{2\pi}\frac{-e_3 + e_4 b - e_5 b^2}{\sqrt{a_3}}.
\end{aligned} \quad (96)$$

On inversion we obtain:

$$\begin{aligned}
G_{33} = &\int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12}e^{-c(t-\tau)} + P_{13})\frac{e^{-\eta_d \tau}}{\sqrt{\tau^2 - r^2/c_d^2}}\\
&\times \cosh(\xi_d\sqrt{\tau^2 - r^2/c_d^2})\,\mathrm{d}\tau\, H(t - r/c_d) + P_{14}\frac{e^{-\eta_d t}}{\sqrt{t^2 - r^2/c_d^2}}\\
&\times \cosh(\xi_d\sqrt{t^2 - r^2/c_d^2})H(t - r/c_d) + \int_{r/c_p}^{t} (P_{21}e^{-b(t-\tau)} + P_{22}e^{-c(t-\tau)} + P_{23})\\
&\times \frac{e^{-\eta_p \tau}}{\sqrt{\tau^2 - r^2/c_p^2}}\cosh(\xi_p\sqrt{\tau^2 - r^2/c_p^2})\,\mathrm{d}\tau\, H(t - r/c_p)\\
&+ P_{24}\frac{e^{-\eta_p t}}{\sqrt{t^2 - r^2/c_p^2}}\cosh(\xi_p\sqrt{t^2 - r^2/c_p^2})H(t - r/c_p),
\end{aligned} \quad (97)$$

where

$$P_{11} = \frac{f_4}{c-b}$$

$$P_{12} = f_3 - \frac{f_4}{c-b}$$

$$P_{13} = f_2$$

$$P_{14} = f_1$$

$$P_{21} = \frac{g_4}{c-b}$$

$$P_{22} = g_3 - \frac{g_4}{c-b}$$

$$P_{23} = g_2$$

$$P_{24} = g_1. \tag{98}$$

Finally eqn (97) confirms our conclusion that no shear wave exists in the field radiated by a fluid source.

## TRANSIENT FUNDAMENTAL SOLUTION (GENERAL CASE-APPROXIMATE GLOBAL FORMULAS)

In the previous section, $\lambda_1$, $\lambda_2$ and $\lambda_3$ are approximated in powers of $1/s$ to a second order polynomial. Thus the two-dimensional transient fundamental solutions obtained constitute a good approximation to the exact results as long as $t$ is small. In order to find the approximate solutions valid for the general case, an alternative approach is pursued. Inspection of eqn (65) along with eqns (29d–g) shows that eqn (65) can be expressed in terms of $s/((1/\kappa m)+s)$, which, obviously, is a nondimensional functional parameter. We observe that $|s/((1/\kappa m)+s)| < 1$ holds for all $s$ and also more interestingly with $s$ decreasing (or $t$ increasing) we would have $|s/((1/\kappa m)+s)| \ll 1$. Thus we are justified in expressing $\lambda_1^2 - \lambda_2^2$ to the first order of approximation in terms of $s/((1/\kappa m)+s)$; a detailed account of the approximation technique will be given in a future paper. After some algebra, we finally arrive at:

$$\lambda_1^2 - \lambda_2^2 \approx cs(s+b), \tag{99}$$

where

$$c = \frac{m(\lambda+2\mu+\alpha^2 Q)}{Q(\lambda+2\mu)} - \frac{\rho+2\alpha\rho_f}{\lambda+2\mu+\alpha^2 Q} + \frac{Q\alpha^2(\rho-2\alpha\rho_f)}{(\lambda+2\mu)(\lambda+2\mu+\alpha^2 Q)}, \tag{100a}$$

$$cb = \frac{(\lambda+2\mu+\alpha^2 Q)}{Q(\lambda+2\mu)\kappa}. \tag{100b}$$

On substitution from eqn (99) and (29d–f) into eqns (66) and (29c), we obtain:

$$\lambda_{1,2} = \frac{1}{c_{d,p}}\sqrt{(s+\eta_{d,p})^2 - \xi_{d,p}^2} \tag{101}$$

where

$$c_p = \sqrt{\frac{\lambda+2\mu+\alpha^2 Q}{\rho}}, \tag{102a}$$

$$c_d = \sqrt{\frac{\lambda+2\mu+\alpha^2 Q}{\rho}}\sqrt{\frac{1}{\frac{m}{\rho}\frac{\lambda+2\mu+\alpha^2 Q}{Q} + \frac{\lambda+2\mu+\alpha^2 Q}{\lambda+2\mu}\frac{\alpha^2 m - 2\alpha\rho_f}{\rho} + \frac{\alpha^2 Q}{\lambda+2\mu}}}, \tag{102b}$$

$$\eta_p = \xi_p \approx 0, \tag{102c}$$

$$\eta_d = \xi_d = \frac{1}{2\kappa m} \frac{\lambda+2\mu+\alpha^2 Q}{\lambda+2\mu+\alpha^2 Q - 2\alpha Q \frac{\rho_f}{m} + \alpha^2 Q \frac{\rho}{m} \frac{Q}{\lambda+2\mu+\alpha^2 Q}}, \tag{102d}$$

which can also be expressed as:

$$\frac{1}{c_{p,d}^2} = \frac{1}{2}\left(a_6 \mp \frac{a_4}{2a_7}\right), \tag{103a}$$

$$\eta_d = \xi_d = \frac{1}{\kappa} \frac{a_7}{a_6 + \frac{a_4}{2a_7}}, \tag{103b}$$

$$\eta_p = \xi_p \approx 0, \tag{103c}$$

where $a_4$, $a_6$, $a_7$ are defined in eqns (68) and (72). Similarly, we deduce that:

$$\lambda_3 = \frac{1}{c_s}\left[s + \frac{\eta_s s}{\varepsilon \eta_s + s}\right], \tag{104}$$

where

$$c_s = \frac{1}{1 - \frac{1}{2}\frac{\rho_f^2}{\rho m}} \sqrt{\frac{\mu}{\rho}}, \tag{105a}$$

$$\eta_s = \frac{1}{2\kappa m} \frac{\rho_f^2}{m\rho - \frac{1}{2}\rho_f^2}, \tag{105b}$$

$$\varepsilon = \frac{2\rho m}{\rho_f^2} - 1. \tag{105c}$$

Again $c_d$, $c_p$, $c_s$ denote velocities of the diffusive wave, pressure wave and shear wave respectively, while $\eta_d$, $\eta_p$, $\eta_s$ denote dissipation factors of the corresponding waves. Interestingly in this case $\eta_p$ is approximately zero, which strongly confirmed Biot's finding that the waves of the first kind (pressure wave) are true waves, their dissipation is practically negligible.

Now with the help of these ingenious approximations for $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_1^2 - \lambda_2^2$, together with available Laplace transform formulas, we start to investigate the transient Green's function, i.e. the fundamental solutions for general case.

*Function $G_{ij}$*

We reintroduce for further consideration eqn (76) which is a transform domain exact solution well posed for approximation. This expression takes a much simpler form if we make the substitution $\lambda_1^2 - \lambda_2^2 = cs(s+b)$, $\lambda_2 = s/c_p$. In this case:

$$\begin{aligned}\tilde{G}_{ij} = {} & \left(f_1 \frac{1}{s} + f_2 \frac{1}{s+b} + f_3 \frac{1}{s(s+b)}\right) K_0(\lambda_1 r) + \left(f_4 \frac{1}{s} + f_5 \frac{1}{s+b} + f_6 \frac{1}{s(s+b)}\right) \frac{1}{\lambda_1} K_1(\lambda_1 r) \\ & + \left(g_1 \frac{1}{s} + g_2 \frac{1}{s+b} + g_3 \frac{1}{s(s+b)}\right) K_0(\lambda_2 r) \\ & + \left(g_4 \frac{1}{s^2} + g_5 \frac{1}{s(s+b)} + g_6 \frac{1}{s^2(s+b)}\right) K_1(\lambda_2 r) + h_1 \frac{1}{s} K_0(\lambda_3 r) + h_2 \frac{1}{s} \frac{1}{\lambda_3} K_1(\lambda_3 r), \end{aligned} \tag{106}$$

where:

$$
\begin{aligned}
f_1 &= -B_{ij}e_{31} \quad g_1 = B_{ij}e_{32} \quad h_1 = -\frac{B_{ij}}{\mu}+C_{ij} \\
f_2 &= -\frac{B_{ij}e_2}{c} \quad g_2 = \frac{B_{ij}e_2}{c} \quad h_2 = -\frac{A_{ij}}{\mu} \\
f_3 &= -\frac{B_{ij}e_1}{c} \quad g_3 = \frac{B_{ij}e_1}{c} \\
f_4 &= -A_{ij}e_{31} \quad g_4 = A_{ij}e_{32}c_p \\
f_5 &= -\frac{A_{ij}e_2}{c} \quad g_5 = \frac{A_{ij}e_2c_p}{c} \\
f_6 &= -\frac{A_{ij}e_1}{c} \quad g_6 = \frac{A_{ij}e_1c_p}{c}.
\end{aligned}
\tag{107}
$$

Making use of the theorem on convolution and taking into consideration eqns (102), (104), after an inverse Laplace transformation performed on eqn (106), along with applying the rule for integration by parts, we find that:

$$
\begin{aligned}
G_{ij} = &\int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)}+P_{12})\frac{e^{-\eta_d\tau}}{\sqrt{\tau^2-r^2/c_d^2}}\cosh\left(\eta_d\sqrt{\tau^2-r^2/c_d^2}\right)\mathrm{d}\tau\, H(t-r/c_d) \\
&+\int_{r/c_d}^{t}(P_{21}e^{-b(t-\tau)}+P_{22})e^{-\eta_d\tau}\sinh\left(\eta_d\sqrt{\tau^2-r^2/c_d^2}\right)\mathrm{d}\tau\, H(t-r/c_d) \\
&+\int_{r/c_p}^{t}\left(P_{31}\frac{1}{\sqrt{\tau^2-r^2/c_p^2}}+P_{32}\sqrt{\tau^2-r^2/c_p^2}\right)e^{-b(t-\tau)}\,\mathrm{d}\tau\, H(t-r/c_p) \\
&+P_{42}t\sqrt{t^2-r^2/c_p^2}H(t-r/c_p)+P_{41}\ln\frac{(t+\sqrt{t^2-r^2/c_p^2})c_p}{r}H(t-r/c_p) \\
&+\int_{r/c_s}^{t}\int_{r/c_s}^{\tau_2}\left(P_{51}\frac{1}{\sqrt{\tau_1^2-r^2/c_s^2}}+P_{52}\sqrt{\tau_1^2-r^2/c_s^2}\right)\sqrt{\frac{\tau_1}{\tau_2-\tau_1}}e^{-\varepsilon\eta_s(\tau_2-\tau_1)}e^{-\eta_s\tau_1} \\
&\times I_1\left(2\eta_s\sqrt{\varepsilon\tau_1(\tau_2-\tau_1)}\right)\mathrm{d}\tau_1\,\mathrm{d}\tau_2\, H(t-r/c_s) \\
&+\int_{r/c_s}^{t}\left(P_{61}\frac{1}{\sqrt{\tau^2-r^2/c_s^2}}+P_{62}\sqrt{\tau^2-r^2/c_s^2}\right)e^{-\eta_s\tau}\,\mathrm{d}\tau\, H(t-r/c_s)
\end{aligned}
\tag{108}
$$

where

$$
\begin{aligned}
P_{11} &= f_2-\frac{f_3}{b} & P_{41} &= g_1+\frac{g_3}{b}-\frac{r}{2c_p}\left(g_4+\frac{g_6}{b}\right) \\
P_{12} &= f_1+\frac{f_3}{b} & P_{42} &= \frac{c_p}{2r}\left(g_4+\frac{g_6}{b}\right) \\
P_{21} &= \left(f_5-\frac{f_6}{b}\right)\frac{c_d^2}{r\eta_d} & P_{51} &= \eta_s\sqrt{\varepsilon}h_1 \\
P_{22} &= \left(f_4+\frac{f_6}{b}\right)\frac{c_d^2}{r\eta_d} & P_{52} &= \eta_s\sqrt{\varepsilon}\frac{c_s^2}{r}h_2 \\
P_{31} &= g_2-\frac{g_3}{b} & P_{61} &= h_1 \\
P_{32} &= \left(g_5-\frac{g_6}{b}\right)\frac{c_p}{r} & P_{62} &= \frac{c_s^2}{r}h_2.
\end{aligned}
\tag{109}
$$

In eqn (108), the first portion of terms (wave speed $c_d$) is highly attenuated wave, the second portion (wave speed $c_p$) is the true wave and the third portion (wave speed $c_s$) is shear wave.

*Function $G_{3i}$*

Now we bring eqn (83) up again for discussion. Taking into account $\lambda_1^2-\lambda_2^2 \approx cs(s+b)$, $\lambda_2 \approx s/c_p$, we see that eqn (83) can be written in the alternative form:

$$\tilde{G}_{3i} = \left(f_1 s+f_2+f_3\frac{1}{s+b}\right)\frac{1}{\lambda_1}K_1(\lambda_1 r)+\left(g_1+g_2\frac{1}{s}+g_3\frac{1}{s(s+b)}\right)K_1\left(\frac{r}{c_p}s\right), \tag{110}$$

where

$$f_1 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_1-\frac{e_5}{c}\right)$$

$$f_2 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_2+\frac{-e_4+e_5 b}{c}\right)$$

$$f_3 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{-e_3+e_4 b-e_5 b^2}{c}$$

$$g_1 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_1+\frac{e_5}{c}\right)c_p$$

$$g_2 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\left(-e_2+\frac{e_4-e_5 b}{c}\right)c_p$$

$$g_3 = \frac{1}{2\pi(\lambda+2\mu)}\frac{x_i}{r}\frac{(e_3-e_4 b+e_5 b^2)c_p}{c}. \tag{111}$$

Now it is easily shown that

$$\begin{aligned} G_{3i} = {} & P_{11}e^{-\eta_d t}\sinh\left(\eta_d\sqrt{t^2-r^2/c_d^2}\right)H(t-r/c_d)-P_{11}e^{-\eta_d t}\cosh\left(\eta_d\sqrt{t^2-r^2/c_d^2}\right) \\ & \times\frac{t}{\sqrt{t^2-r^2/c_d^2}}H(t-r/c_d)+P_{12}e^{-\eta_d t}\sinh\left(\eta_d\sqrt{t^2-r^2/c_d^2}\right)H(t-r/c_d) \\ & +\int_{r/c_d}^{t}P_{13}e^{-b(t-\tau)}e^{-\eta_d\tau}\sinh\left(\eta_d\sqrt{\tau^2-r^2/c_d^2}\right)\mathrm{d}\tau\,H(t-r/c_d)+\int_{r/c_p}^{t}P_{21}e^{-b(t-\tau)} \\ & \times\frac{\tau}{\sqrt{\tau^2-r^2/c_p^2}}\mathrm{d}\tau\,H(t-r/c_p)+P_{22}\sqrt{t^2-r^2/c_p^2}H(t-r/c_p) \\ & +P_{23}\frac{t}{\sqrt{t^2-r^2/c_p^2}}H(t-r/c_p), \end{aligned} \tag{112}$$

where

$$P_{11} = -f_1\frac{c_d^2}{r}$$

$$P_{12} = f_2\frac{c_d^2}{\eta_d r}$$

$$P_{13} = f_3\frac{c_d^2}{\eta_d r}$$

$$P_{21} = -\frac{g_3}{b}\frac{c_p}{r}$$

$$P_{22} = \left(g_2 + \frac{g_3}{b}\right)\frac{c_p}{r}$$

$$P_{23} = g_1\frac{c_p}{r}. \tag{113}$$

Equation (112) clearly indicates that in the fluid pressure field only two dilatational waves propagate from the line force.

*Function $G_{i3}$*

We deduce immediately from the identity $\tilde{G}_{i3} = (1/s)\tilde{G}_{3i}$ and eqn (110) that:

$$\tilde{G}_{i3} = \left(f_1 + f_2\frac{1}{s} + f_3\frac{1}{s(s+b)}\right)\frac{1}{\lambda_1}K_1(\lambda_1 r) + \left(g_1\frac{1}{s} + g_2\frac{1}{s^2} + g_3\frac{1}{s^2(s+b)}\right)K_1(\lambda_2 r), \tag{114}$$

where $f_1, f_2, f_3, g_1, g_2, g_3$ have the same values as in eqn (111).

The desired transient Green's function can be easily obtained by applying Laplace inversion:

$$\begin{aligned} G_{i3} = {} & \int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12})e^{-\eta_d\tau}\sinh\left(\eta_d\sqrt{\tau^2 - r^2/c_d^2}\right)\mathrm{d}\tau\, H(t - r/c_d) \\ & + P_{13}e^{-\eta_d t}\sinh\left(\eta_d\sqrt{t^2 - r^2/c_d^2}\right)H(t - r/c_d) + \int_{r/c_p}^{t} P_{21}e^{-b(t-\tau)} \\ & \times \frac{\tau}{\sqrt{\tau^2 - r^2/c_p^2}}\,\mathrm{d}\tau\, H(t - r/c_p) + (P_{22} + P_{23}t)\sqrt{t^2 - r^2/c_p^2}H(t - r/c_p) \\ & + P_{24}\ln\frac{(t + \sqrt{t^2 - r^2/c_p^2})c_p}{r}H(t - r/c_p) \end{aligned} \tag{115}$$

where

$$P_{11} = -\frac{f_3}{b}\frac{c_d^2}{\eta_d r}$$

$$P_{12} = \left(f_2 + \frac{f_3}{b}\right)\frac{c_d^2}{\eta_d r}$$

$$P_{13} = f_1\frac{c_d^2}{\eta_d r}$$

$$P_{21} = \frac{g_3}{b^2}\frac{c_p}{r}$$

$$P_{22} = \left(g_1 - \frac{g_3}{b^2}\right)\frac{c_p}{r}$$

$$P_{23} = \frac{1}{2}\left(g_2 + \frac{g_3}{b}\right)\frac{c_p}{r}$$

$$P_{24} = \frac{-r}{2c_p}\left(g_2 + \frac{g_3}{b}\right). \tag{116}$$

The contribution of shear wave is zero in the solid displacement generated by a unit line fluid source.

*Function* $G_{33}$

Finally, with the help of eqn (93) and eqn (99), we introduce the following expression:

$$\tilde{G}_{33} = \left(f_1 + f_2\frac{1}{s} + f_3\frac{1}{s+b} + f_4\frac{1}{s(s+b)}\right)K_0(\lambda_1 r) + \left(g_1 + g_2\frac{1}{s} + g_3\frac{1}{s+b} + g_4\frac{1}{s(s+b)}\right)K_0(\lambda_2 r), \quad (117)$$

where

$$f_1 = \frac{1}{2\pi}\left(e_1 + \frac{e_5}{c}\right)$$

$$f_2 = \frac{1}{2\pi}e_2$$

$$f_3 = \frac{1}{2\pi}\frac{e_4 - e_5 b}{c}$$

$$f_4 = \frac{1}{2\pi}\frac{e_3}{c}$$

$$g_1 = \frac{1}{2\pi}\left(e_1 - \frac{e_5}{c}\right)$$

$$g_2 = \frac{1}{2\pi}e_2$$

$$g_3 = \frac{1}{2\pi}\frac{-e_4 + e_5 b}{c}$$

$$g_4 = -\frac{1}{2\pi}\frac{e_3}{c}, \quad (118)$$

which are now inverted to yield:

$$G_{33} = \int_{r/c_d}^{t} (P_{11}e^{-b(t-\tau)} + P_{12})\frac{e^{-\eta_d\tau}}{\sqrt{\tau^2 - r^2/c_d^2}}\cosh\left(\eta_d\sqrt{\tau^2 - r^2/c_d^2}\right)\mathrm{d}\tau\, H(t - r/c_d)$$
$$+ P_{13}\frac{e^{-\eta_d t}}{\sqrt{t^2 - r^2/c_d^2}}\cosh\left(\eta_d\sqrt{t^2 - r^2/c_d^2}\right)H(t - r/c_d) + P_{21}\int_{r/c_p}^{t} e^{-b(t-\tau)}$$
$$\times \frac{1}{\sqrt{\tau^2 - r^2/c_p^2}}\mathrm{d}\tau\, H(t - r/c_p) + P_{22}\ln\frac{(t + \sqrt{t^2 - r^2/c_p^2})c_p}{r}H(t - r/c_p)$$
$$+ P_{23}\frac{1}{\sqrt{t^2 - r^2/c_p^2}}H(t - r/c_p), \quad (119)$$

where

$$P_{11} = f_3 - \frac{f_4}{b}$$

$$P_{12} = f_2 + \frac{f_4}{b}$$

$$P_{13} = f_1$$

$$P_{21} = g_3 - \frac{g_4}{b}$$

$$P_{22} = g_2 + \frac{g_4}{b}$$

$$P_{23} = g_1. \tag{120}$$

The nonexistence of shear wave in the pressure field is again justified.

## RECIPROCAL THEOREM AND INTEGRAL REPRESENTATION

One of the basic ingredients in the BEM is an appropriate reciprocal theorem, which provides a mathematical basis for a direct boundary element formulation. The Betti's reciprocal theorem (1844) is one of the most interesting and most useful theorems in the theory of elasticity. Its elastodynamic counterpart was obtained by Graffi (1946–1947) for bounded regions and extended by Wheeler and Sternberg (1968) to unbounded regions. The considered theorem was extended also to problems of dynamic thermoelasticity by Ionescu–Cazimir (1964). This section is dedicated to extend the theorem to dynamic poroelasticity, thereby leading to a time domain boundary integral representation for full dynamic poroelasticity.

*Reciprocal theorem*

Let $V$ be an interior or exterior domain in an Euclidian space $\mathbf{R}^n$ ($n = 2, 3$) bounded by $S$. The unit outward normal vector $\mathbf{n} = (n_i)$ is defined at a point on $S$. Now consider the poroelasticity region $V$ with the boundary $S$ subject to two different systems of external loading:

$$\{f_i^{(\alpha)}, t_i^{(\alpha)}, \gamma^{(\alpha)}, p^{(\alpha)}\}, \quad \alpha = 1, 2,$$

where $f_i^{(\alpha)}$ body force, $t_i^{(\alpha)}$ surface traction, $\gamma^{(\alpha)}$ fluid source, $p^{(\alpha)}$ surface pressure. They produce in the body two poroelastic configurations:

$$\{u_i^{(\alpha)}, w_i^{(\alpha)}, p^{(\alpha)}\},$$

and therefore the stresses $\sigma_{ij}(x,t)$ and strain $\varepsilon_{ij}(x,t)$. We assume $\sigma_{ij}$, $\varepsilon_{ij}$ of class $C^{(1)}$ and $u_i(x,t)$, $w_i(x,t)$, $p(x,t)$ of class $C^{(2)}$, all for $x \in V+S$, $t > 0$ and also the homogeneous initial conditions:

$$u_i(x,0) = 0, \quad \dot{u}_i(x,0) = 0, \quad w_i(x,0) = 0, \quad \dot{w}_i(x,0) = 0, \quad p(x,0) = 0, \quad \text{for} \quad x \in V. \tag{121}$$

We apply Laplace transformation with respect to the time $t$ to eqns (1), (3), (4), (5), (7) assuming that the transforms exist. We have, due to the initial conditions indicated in eqn (121),

$$\tilde{\sigma}_{ij,j}^{(\alpha)} + \tilde{f}_i^{(\alpha)} = \rho s^2 \tilde{u}_i^{(\alpha)} + \rho_f s^2 \tilde{w}_i^{(\alpha)} \tag{122}$$

$$s\tilde{w}_{k,k}^{(\alpha)} + s\frac{1}{Q}\tilde{p}^{(\alpha)} + \alpha s \tilde{u}_{k,k}^{(\alpha)} = \tilde{\gamma}^{(\alpha)} \tag{123}$$

$$\tilde{\sigma}_{ij}^{(\alpha)} = (\lambda \tilde{u}_{k,k}^{(\alpha)} - \alpha \tilde{p}^{(\alpha)})\delta_{ij} + \mu(\tilde{u}_{i,j}^{(\alpha)} + \tilde{u}_{j,i}^{(\alpha)}) \tag{124}$$

$$\tilde{p}_{,i}^{(\alpha)} = -\frac{s}{\kappa}\tilde{w}_i^{(\alpha)} - \rho_f s^2 \tilde{u}_i^{(\alpha)} - m s^2 \tilde{w}_i^{(\alpha)} \tag{125}$$

$$\tilde{q}_i^{(\alpha)} = s\tilde{w}_i^{(\alpha)}. \tag{126}$$

Using eqns (124) and (125) one obtains:

$$\tilde{\sigma}_{ij}^{(\alpha)} + \alpha\tilde{p}^{(\alpha)}\delta_{ij} = \lambda\tilde{u}_{k,k}^{(\alpha)}\delta_{ij} + 2\mu\tilde{\varepsilon}_{ij}^{(\alpha)} \tag{127}$$

$$\tilde{w}_{i,i}^{(\alpha)} = \frac{\zeta}{s}(-\delta_{ij}\tilde{p}_{,ij}^{(\alpha)} - \rho_f s^2\tilde{\varepsilon}_{ii}^{(\alpha)}, \tag{128}$$

where $\zeta = 1/(1/\kappa + ms)$.

On the basis of (127), we derive:

$$\tilde{\sigma}_{ij}^{(1)}\tilde{\varepsilon}_{ij}^{(2)} + \alpha\delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}_{ij}^{(2)} = \tilde{\sigma}_{ij}^{(2)}\tilde{\varepsilon}_{ij}^{(1)} + \alpha\delta_{ij}\tilde{p}^{(2)}\tilde{\varepsilon}_{ij}^{(1)}. \tag{129}$$

Integrating (129) over the region $V$, we obtain:

$$\int_V \tilde{\sigma}_{ij}^{(1)}\tilde{\varepsilon}_{ij}^{(2)}\,dV + \int_V \alpha\delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}_{ij}^{(2)}\,dV = \int_V \tilde{\sigma}_{ij}^{(2)}\tilde{\varepsilon}_{ij}^{(1)}\,dV + \int_V \alpha\delta_{ij}\tilde{p}^{(2)}\tilde{\varepsilon}_{ij}^{(1)}\,dV. \tag{130}$$

Using eqn (122), the divergence theorem and the boundary condition $\tilde{t}_i^{(\alpha)} = \tilde{\sigma}_{ij}^{(\alpha)}n_j$, the first term on the left side of eqn (130) can be expressed as:

$$\int_V \tilde{\sigma}_{ij}^{(1)}\tilde{\varepsilon}_{ij}^{(2)}\,dV = \int_S \tilde{t}_i^{(1)}\tilde{u}_i^{(2)}\,dS + \int_V [\tilde{f}_i^{(1)} - \rho s^2\tilde{u}_i^{(1)} + \rho_f s\zeta\tilde{p}_{,i}^{(1)} + \rho_f^2 s^3\zeta\tilde{u}_i^{(1)}]\tilde{u}_i^{(2)}\,dV. \tag{131}$$

A similar expression is obtained for the first integral on the right side in eqn (130):

$$\int_V \tilde{\sigma}_{ij}^{(2)}\tilde{\varepsilon}_{ij}^{(1)}\,dV = \int_S \tilde{t}_i^{(2)}\tilde{u}_i^{(1)}\,dS + \int_V [\tilde{f}_i^{(2)} - \rho s^2\tilde{u}_i^{(2)} + \rho_f s\zeta\tilde{p}_{,i}^{(2)} + \rho_f^2 s^3\zeta\tilde{u}_i^{(2)}]\tilde{u}_i^{(1)}\,dV. \tag{132}$$

By introducing expressions (131), (132) into (130) and subtracting, one obtains:

$$\int_V \tilde{f}_i^{(1)}\tilde{u}_i^{(2)}\,dV + \int_S \tilde{t}_i^{(1)}\tilde{u}_i^{(2)}\,dS + \int_V [\rho_f s\zeta\tilde{p}_{,i}^{(1)}\tilde{u}_i^{(2)} + \alpha\delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}_{ij}^{(2)}]\,dV$$
$$= \int_V \tilde{f}_i^{(2)}\tilde{u}_i^{(1)}\,dV + \int_S \tilde{t}_i^{(2)}\tilde{u}_i^{(1)}\,dS + \int_V [\rho_f s\zeta\tilde{p}_{,i}^{(2)}\tilde{u}_i^{(1)} + \alpha\delta_{ij}\tilde{p}^{(2)}\tilde{\varepsilon}_{ij}^{(1)}]\,dV. \tag{133}$$

Substitution of (128) into (123) yields:

$$\zeta[\delta_{ij}\tilde{p}_{,ij}^{(\alpha)} + \rho_f s^2\delta_{ij}\tilde{u}_{i,j}^{(\alpha)}] - s\frac{1}{Q}\tilde{p}^{(\alpha)} - \alpha s\delta_{ij}\tilde{u}_{i,j}^{(\alpha)} = -\tilde{\gamma}^{(\alpha)}. \tag{134}$$

Using the divergence theorem and the boundary condition, one obtains:

$$\zeta\int_V \delta_{ij}\tilde{p}^{(1)}\tilde{p}_{,ij}^{(2)}\,dV = \zeta\int_V (\delta_{ij}\tilde{p}^{(1)}\tilde{p}_{,i}^{(2)})_{,j}\,dV - \zeta\int_V \delta_{ij}\tilde{p}_{,j}^{(1)}\tilde{p}_{,i}^{(2)}\,dV$$
$$= \zeta\int_S (\delta_{ij}\tilde{p}^{(1)}\tilde{p}_{,i}^{(2)})n_j\,dS - \zeta\int_V \delta_{ij}\tilde{p}_{,j}^{(1)}\tilde{p}_{,i}^{(2)}\,dV. \tag{135}$$

Making use of (134), one obtains:

$$\zeta\int_V \delta_{ij}\tilde{p}^{(1)}\tilde{p}^{(2)}_{,ij}\, V = -\zeta\int_V \rho_f s^2 \delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}^{(2)}_{ij}\,\mathrm{d}V + \alpha s\int_V \delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}^{(2)}_{ij}\,\mathrm{d}V + \frac{1}{Q}s\int_V \tilde{p}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V$$

$$-\int_V \tilde{\gamma}^{(2)}\tilde{p}^{(1)}\,\mathrm{d}V = s(\alpha-\zeta\rho_f s)\int_V \delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}^{(2)}_{ij}\,\mathrm{d}V + \frac{1}{Q}s\int_V \tilde{p}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V \quad -\int_V \tilde{\gamma}^{(2)}\tilde{p}^{(1)}\,\mathrm{d}V. \tag{136}$$

By equating (135) and (136), one obtains:

$$\int_V \delta_{ij}\tilde{p}^{(1)}\tilde{\varepsilon}^{(2)}_{ij}\,\mathrm{d}V = \frac{1}{s(\alpha-\zeta\rho_f s)}\left\{-\frac{1}{Q}s\int_V \tilde{p}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V + \int_V \tilde{\gamma}^{(2)}\tilde{p}^{(1)}\,\mathrm{d}V\right.$$
$$\left. +\zeta\int_S \delta_{ij}\tilde{p}^{(1)}\tilde{p}^{(2)}_{,i}n_j\,\mathrm{d}S - \zeta\int_V \delta_{ij}\tilde{p}^{(1)}_{,j}\tilde{p}^{(2)}_{,i}\,\mathrm{d}V\right\}. \tag{137}$$

The third term at the left side of eqn (133) can be expressed as:

$$\int_V \rho_f s\zeta\tilde{p}^{(1)}_{,i}\tilde{u}^{(2)}_i\,\mathrm{d}V = \int_V [\rho_f s\zeta\tilde{p}^{(1)}\tilde{u}^{(2)}_i]_{,i}\,\mathrm{d}V - \int_V [\rho_f s\zeta\tilde{p}^{(1)}\tilde{u}^{(2)}_{i,i}]\,\mathrm{d}V$$
$$= \int_S \rho_f s\zeta\tilde{p}^{(1)}\tilde{u}^{(2)}_i n_i\,\mathrm{d}S - \int_V \rho_f s\zeta\tilde{p}^{(1)}\tilde{\varepsilon}^{(2)}_{ii}\,\mathrm{d}V$$
$$= \rho_f s\zeta\int_S \tilde{p}^{(1)}\tilde{u}^{(2)}_i n_i\,\mathrm{d}S - \rho_f s\zeta\int_V \tilde{p}^{(1)}\delta_{ij}\tilde{\varepsilon}^{(2)}_{ij}\,\mathrm{d}V, \tag{138}$$

and, analogously:

$$\int_V \delta_{ij}\tilde{p}^{(2)}\tilde{\varepsilon}^{(1)}_{ij}\,\mathrm{d}V = \frac{1}{s(\alpha-\zeta\rho_f s)}\left\{-\frac{1}{Q}s\int_V \tilde{p}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V + \int_V \tilde{\gamma}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V\right.$$
$$\left. +\zeta\int_S \delta_{ij}\tilde{p}^{(2)}\tilde{p}^{(1)}_{,i}n_j\,\mathrm{d}S - \zeta\int_V \delta_{ij}\tilde{p}^{(2)}_{,j}\tilde{p}^{(1)}_{,i}\,\mathrm{d}V\right\}, \tag{139}$$

$$\int_V \rho_f s\zeta\tilde{p}^{(2)}_{,i}\tilde{u}^{(1)}_i\,\mathrm{d}V = \rho_f s\zeta\int_S \tilde{p}^{(2)}\tilde{u}^{(1)}_i n_i\,\mathrm{d}S - \rho_f s\zeta\int_V \tilde{p}^{(2)}\delta_{ij}\tilde{\varepsilon}^{(1)}_{ij}\,\mathrm{d}V. \tag{140}$$

When these expressions [eqns (137)–(140)] are substituted into eqn (133), we see that a number of volume integrals cancel each other. Thus one obtains:

$$\int_V \tilde{f}^{(1)}_i\tilde{u}^{(2)}_i\,\mathrm{d}V + \int_S \tilde{t}^{(1)}_i\tilde{u}^{(2)}_i\,\mathrm{d}S + \rho_f s\zeta\int_S \tilde{p}^{(1)}\tilde{u}^{(2)}_i n_i\,\mathrm{d}S$$
$$+\frac{1}{s}\int_V \tilde{\gamma}^{(2)}\tilde{p}^{(1)}\,\mathrm{d}V + \frac{\zeta}{s}\int_S (\delta_{ij}\tilde{p}^{(1)}\tilde{p}^{(2)}_{,i})n_j\,\mathrm{d}S$$
$$= \int_V \tilde{f}^{(2)}_i\tilde{u}^{(1)}_i\,\mathrm{d}V + \int_S \tilde{t}^{(2)}_i\tilde{u}^{(1)}_i\,\mathrm{d}S + \rho_f s\zeta\int_S \tilde{p}^{(2)}\tilde{u}^{(1)}_i n_i\,\mathrm{d}S$$
$$+\frac{1}{s}\int_V \tilde{\gamma}^{(1)}\tilde{p}^{(2)}\,\mathrm{d}V + \frac{\zeta}{s}\int_S (\delta_{ij}\tilde{p}^{(2)}\tilde{p}^{(1)}_{,i})n_j\,\mathrm{d}S. \tag{141}$$

Finally, taking into account eqns (125), (126) we have normal flux boundary condition

$$\tilde{q}^{(\alpha)} = s\tilde{w}_i^{(\alpha)} n_i = \zeta(-\tilde{p}_{,i}^{(\alpha)} - \rho_f s^2 \tilde{u}_i^{(\alpha)}) n_i, \tag{142}$$

we arrive at:

$$\int_S s\tilde{t}_i^{(1)}\tilde{u}_i^{(2)}\,dS - \int_S \tilde{p}^{(1)}\tilde{q}^{(2)}\,dS + \int_V s\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}\,dV - \int_V \tilde{\gamma}^{(1)}\tilde{p}^{(2)}\,dV$$
$$= \int_S s\tilde{t}_i^{(2)}\tilde{u}_i^{(1)}\,dS - \int_S \tilde{p}^{(2)}\tilde{q}^{(1)}\,dS + \int_V s\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}\,dV - \int_V \tilde{\gamma}^{(2)}\tilde{p}^{(1)}\,dV. \tag{143}$$

This is the reciprocal theorem expressed in Laplace transform domain. If we remove the tilde and consider the variables to be in the real-time domain, then it becomes Betti's reciprocal relation for dynamic poroelasticity in time domain.

Since the inverse transform of the product of two functions is the convolution of the inverse, we obtain the desired reciprocal theorem:

$$\int_S [t_i^{(1)} * \dot{u}_i^{(2)} - t_i^{(2)} * \dot{u}_i^{(1)} - p^{(1)} * q^{(2)} + p^{(2)} * q^{(1)}]\,dS$$
$$+ \int_V [f_i^{(1)} * \dot{u}_i^{(2)} - f_i^{(2)} * \dot{u}_i^{(1)} - \gamma^{(1)} * p^{(2)} + \gamma^{(2)} * p^{(1)}]\,dV = 0, \tag{144}$$

where the symbol $*$ stands for the convolution product, a superimposed dot indicates differentiation with respect to time.

$$L^{-1}\{\tilde{f}_1(s)\tilde{f}_2(s)\} = f_1 * f_2 = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_2(\tau)f_1(t-\tau)\,d\tau. \tag{145}$$

We have assumed in our deriving of eqn (144) that on the boundary $S$ tractions $t_i^{(\alpha)}$ are prescribed as well as the pressure $p^{(\alpha)}$. It follows from the structure of eqns (143), (144) that we may prescribe on boundary $S$ also the displacements and the fluid flux. Equations (143), (144) are also valid for mixed boundary conditions such as:

$$\int_S [\dot{t}_i^{(1)} * u_i^{(2)} + q^{(1)} * p^{(2)} - \dot{u}_i^{(1)} * t_i^{(2)} - p^{(1)} * q^{(2)}]\,dS$$
$$+ \int_V [\dot{f}_i^{(1)} * u_i^{(2)} + f_{i0}^{(1)} u_i^{(2)} + p^{(1)} * \gamma^{(2)} - \dot{u}_i^{(1)} * f_i^{(2)} - \gamma^{(1)} * p^{(2)}]\,dV = 0, \tag{146}$$

where $f_{i0}^{(1)} = f_i^{(1)}(x, 0)$. It should be mentioned that theorem (146) is more desirable than theorem (144), since the former will lead to a boundary integral representation with displacement, traction, fluid pore pressure and flux as primary quantities. Further we extend the theorem to include the non-zero initial conditions:

$$u_i(x,0) = u_{i0}, \quad \dot{u}_i(x,0) = \dot{u}_{i0}, \quad w_i(x,0) = w_{i0},$$
$$\dot{w}_i(x,0) = \dot{w}_{i0} = q_i(x,0) = q_{i0}, \quad p(x,0) = p_0. \tag{147}$$

Following the calculations mentioned and with the aid of the Laplace transform formulas for the first and second time derivative with nonzero initial conditions, one obtains:

$$\int_S \tilde{t}_i^{(1)}\tilde{u}_i^{(2)}\,\mathrm{d}S-\int_S \tilde{p}^{(1)}\frac{1}{s}\tilde{q}^{(2)}\,\mathrm{d}S+\int_V [\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}+\rho(u_{i0}^{(1)}s\tilde{u}_i^{(2)}+\dot{u}_{i0}^{(1)}\tilde{u}_i^{(2)})+\rho_f q_{i0}^{(1)}\tilde{u}_i^{(2)}]\,\mathrm{d}V$$
$$-\int_V \frac{1}{s}\left[\tilde{\gamma}^{(1)}+\frac{1}{Q}p_0^{(1)}+\alpha u_{i0,i}^{(1)}\right]\tilde{p}^{(2)}\,\mathrm{d}V+\int_V\left[\rho_f u_{i0}^{(1)}\tilde{q}_i^{(2)}+(\rho_f\dot{u}_{i0}^{(1)}+mq_{i0}^{(1)})\frac{1}{s}\tilde{q}_i^{(2)}\right]\mathrm{d}V$$
$$=\int_S \tilde{t}_i^{(2)}\tilde{u}_i^{(1)}\,\mathrm{d}S-\int_S \tilde{p}^{(2)}\frac{1}{s}\tilde{q}^{(1)}\,\mathrm{d}S+\int_V [\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}+\rho(u_{i0}^{(2)}s\tilde{u}_i^{(1)}+\dot{u}_{i0}^{(2)}\tilde{u}_i^{(1)})+\rho_f q_{i0}^{(2)}\tilde{u}_i^{(1)}]\,\mathrm{d}V$$
$$-\int_V \frac{1}{s}\left[\tilde{\gamma}^{(2)}+\frac{1}{Q}p_0^{(2)}+\alpha u_{i0,i}^{(2)}\right]\tilde{p}^{(1)}\,\mathrm{d}V+\int_V\left[\rho_f u_{i0}^{(2)}\tilde{q}_i^{(1)}+(\rho_f\dot{u}_{i0}^{(2)}+mq_{i0}^{(2)})\frac{1}{s}\tilde{q}_i^{(1)}\right]\mathrm{d}V. \tag{148}$$

It remains to invert eqn (148) to the time domain:

$$\int_S t_i^{(1)}*u_i^{(2)}\,\mathrm{d}S-\int_S p^{(1)}*Q_n^{(2)}\,\mathrm{d}S+\int_V [f_i^{(1)}*u_i^{(2)}+\rho(u_{i0}^{(1)}\dot{u}_i^{(2)}+\dot{u}_{i0}^{(1)}u_i^{(2)})+\rho_f q_{i0}^{(1)}u_i^{(2)}]\,\mathrm{d}V$$
$$-\int_V\left[\Gamma^{(1)}*p^{(2)}+\left(\frac{1}{Q}p_0^{(1)}+\alpha u_{i0,i}^{(1)}\right)*p^{(2)}\right]\mathrm{d}V+\int_V [\rho_f u_{i0}^{(1)}q_i^{(2)}+(\rho_f\dot{u}_{i0}^{(1)}+mq_{i0}^{(1)})Q_i^{(2)}]\,\mathrm{d}V$$
$$=\int_S t_i^{(2)}*u_i^{(1)}\,\mathrm{d}S-\int_S p^{(2)}*Q_n^{(1)}\,\mathrm{d}S+\int_V [f_i^{(2)}*u_i^{(1)}+\rho(u_{i0}^{(2)}\dot{u}_i^{(1)}+\dot{u}_{i0}^{(2)}u_i^{(1)})+\rho_f q_{i0}^{(2)}u_i^{(1)}]\,\mathrm{d}V$$
$$-\int_V\left[\Gamma^{(2)}*p^{(1)}+\left(\frac{1}{Q}p_0^{(2)}+\alpha u_{i0,i}^{(2)}\right)*p^{(1)}\right]\mathrm{d}V+\int_V\left[\rho_f u_{i0}^{(2)}q_i^{(1)}+(\rho_f\dot{u}_{i0}^{(2)}+mq_{i0}^{(2)})Q_i^{(1)}\right]\mathrm{d}V, \tag{149}$$

in which $u_{i0,i}^{(\alpha)}=(\partial u_{i0}^{(\alpha)}/\partial x_i)$, we have also introduced the following notation:

$$Q_i^{(\alpha)}=\int_0^t q_i^{(\alpha)}(\mathbf{x},\tau)\,\mathrm{d}\tau$$
$$Q_n^{(\alpha)}=Q_i^{(\alpha)}n_i$$
$$\Gamma^{(\alpha)}=\int_0^t \gamma^{(\alpha)}\,\mathrm{d}\tau, \tag{150}$$

where $Q_i^{(\alpha)}$ and $Q_n^{(\alpha)}$ are the total fluid flow through the unit volume of the media in the $x_i$ direction and through the unit surface area in its outward normal direction, respectively. The counterparts of eqns (148) and (149) for quasistatic case are available in Cleary (1976).

In turn another form of reciprocal theorem with nonzero initial conditons can be obtained by substituting equation $(1/s)\tilde{q}^{(\alpha)}=\tilde{w}_i^{(\alpha)}n_i-(1/s)w_{i0}^{(\alpha)}n_i$ into eqn (148), again using the divergence theorem and with reorganization of terms:

$$\int_S \tilde{t}_i^{(1)}\tilde{u}_i^{(2)}\,\mathrm{d}S-\int_S \tilde{p}^{(1)}\tilde{w}_n^{(2)}\,\mathrm{d}S+\int_V [\tilde{f}_i^{(1)}\tilde{u}_i^{(2)}+\rho(u_{i0}^{(1)}s\tilde{u}_i^{(2)}+\dot{u}_{i0}^{(1)}\tilde{u}_i^{(2)})$$
$$+\rho_f(\dot{w}_{i0}^{(1)}\tilde{u}_i^{(2)}+w_{i0}^{(1)}s\tilde{u}_i^{(2)})]\,\mathrm{d}V-\int_V\frac{1}{s}\left[\tilde{\gamma}^{(1)}+\frac{1}{Q}p_0^{(1)}+\alpha u_{i0,i}^{(1)}+w_{i0,i}^{(1)}\right]\tilde{p}^{(2)}\,\mathrm{d}V$$
$$+\int_V\left[\frac{1}{\kappa}w_{i0}^{(1)}\tilde{w}_i^{(2)}+\rho_f(u_{i0}^{(1)}s\tilde{w}_i^{(2)}+\dot{u}_{i0}^{(1)}\tilde{w}_i^{(2)})+m(w_{i0}^{(1)}s\tilde{w}_i^{(2)}+\dot{w}_{i0}^{(1)}\tilde{w}_i^{(2)})\right]\mathrm{d}V$$
$$=\int_S \tilde{t}_i^{(2)}\tilde{u}_i^{(1)}\,\mathrm{d}S-\int_S \tilde{p}^{(2)}\tilde{w}_n^{(1)}\,\mathrm{d}S+\int_V [\tilde{f}_i^{(2)}\tilde{u}_i^{(1)}+\rho(u_{i0}^{(2)}s\tilde{u}_i^{(1)}+\dot{u}_{i0}^{(2)}\tilde{u}_i^{(1)})$$

$$
+\rho_f(\dot{w}_{i0}^{(2)}\tilde{u}_i^{(1)}+w_{i0}^{(2)}s\tilde{u}_i^{(1)})]\,\mathrm{d}V-\int_V \frac{1}{s}\left[\bar{\gamma}^{(2)}+\frac{1}{Q}p_0^{(2)}+\alpha u_{i0,i}^{(2)}+w_{i0,i}^{(2)}\right]\tilde{p}^{(1)}\,\mathrm{d}V
$$

$$
+\int_V\left[\frac{1}{\kappa}w_{i0}^{(2)}\tilde{w}_i^{(1)}+\rho_f(u_{i0}^{(2)}s\tilde{w}_i^{(1)}+\dot{u}_{i0}^{(2)}\tilde{w}_i^{(1)})+m(w_{i0}^{(2)}s\tilde{w}_i^{(1)}+\dot{w}_{i0}^{(2)}\tilde{w}_i^{(1)})\right]\mathrm{d}V \tag{151}
$$

where $w_{i0,i}^{(\alpha)}=(\partial w_{i0}^{(\alpha)})/(\partial x_i)$, $\tilde{w}_n^{(\alpha)}=\tilde{w}_i^{(\alpha)}n_i$. After applying Laplace inversion, the following new form of reciprocal theorem is obtained:

$$
\int_S t_i^{(1)}*u_i^{(2)}\,\mathrm{d}S-\int_S p^{(1)}*w_n^{(2)}\,\mathrm{d}S+\int_V [f_i^{(1)}*u_i^{(2)}+\rho(u_{i0}^{(1)}\dot{u}_i^{(2)}+\dot{u}_{i0}^{(1)}u_i^{(2)})
$$

$$
+\rho_f(w_{i0}^{(1)}\dot{u}_i^{(2)}+\dot{w}_{i0}^{(1)}u_i^{(2)})]\,\mathrm{d}V-\int_V\left[\Gamma^{(1)}+\frac{1}{Q}p_0^{(1)}+\alpha u_{i0,i}^{(1)}+w_{i0,i}^{(1)}\right]*p^{(2)}\,\mathrm{d}V
$$

$$
+\int_V\left[\frac{1}{\kappa}w_{i0}^{(1)}w_i^{(2)}+\rho_f(u_{i0}^{(1)}\dot{w}_i^{(2)}+\dot{u}_{i0}^{(1)}w_i^{(2)})+m(w_{i0}^{(1)}\dot{w}_i^{(2)}+\dot{w}_{i0}^{(1)}w_i^{(2)})\right]\mathrm{d}V
$$

$$
=\int_S t_i^{(2)}*u_i^{(1)}\,\mathrm{d}S-\int_S p^{(2)}*w_n^{(1)}\,\mathrm{d}S+\int_V [f_i^{(2)}*u_i^{(1)}+\rho(u_{i0}^{(2)}\dot{u}_i^{(1)}+\dot{u}_{i0}^{(2)}u_i^{(1)})
$$

$$
+\rho_f(w_{i0}^{(2)}\dot{u}_i^{(1)}+\dot{w}_{i0}^{(2)}u_i^{(1)})]\,\mathrm{d}V-\int_V\left[\Gamma^{(2)}+\frac{1}{Q}p_0^{(2)}+\alpha u_{i0,i}^{(2)}+w_{i0,i}^{(2)}\right]*p^{(1)}\,\mathrm{d}V
$$

$$
+\int_V\left[\frac{1}{\kappa}w_{i0}^{(2)}w_i^{(1)}+\rho_f(u_{i0}^{(2)}\dot{w}_i^{(1)}+\dot{u}_{i0}^{(2)}w_i^{(1)})+m(w_{i0}^{(2)}\dot{w}_i^{(1)}+\dot{w}_{i0}^{(2)}w_i^{(1)})\right]\mathrm{d}V. \tag{152}
$$

Now the volume integrals contain body forces, fluid sources, the initial conditions and viscous forces. If we add another fluid body force vector $F_i$ to the right hand side of the generalized Darcy's law eqn (3), the supplementary terms $F_i^{(1)}*Q_i^{(2)}$ and $F_i^{(2)}*Q_i^{(1)}$ should be added to the third volume integral on the left and right side of eqn (149) separately, while the additional terms $F_i^{(1)}*w_i^{(2)}$ and $F_i^{(2)}*w_i^{(1)}$ will appear in the third volume integral on the left and right side, respectively, in eqn (152).

## INTEGRAL REPRESENTATION

In the reciprocal relation, i.e. eqn (146), we can make the following substitution: (1) Only a point force at $\xi$ in the direction of $x_j$-axis with Heaviside step time function in the infinite region: $f_i^{(1)}(z,t)=\delta(z-\xi)H(t)\delta_{ij}e_j$, $\gamma^{(1)}(z,t)=0$ and $f_i^{(2)}(z,t)=0$, $\gamma^{(2)}(z,t)=0$, consequently $u_i^{(1)}(x,t)=G_{ij}(x-\xi.t)e_j$, $p^{(1)}(x,t)=G_{pj}(x-\xi,t)e_j$, $t_i^{(1)}(x,t)F_{ij}(x-\xi,t)e_j$, $q^{(1)}(x,t)=F_{pj}(x-\xi,t)e_j$; (2) Only a fluid injection at $\xi$ with Dirac delta time function in the infinite region: $f_i^{(1)}(z,t)=0$, $\gamma^{(1)}(z,t)=\delta(z-\xi)\delta(t)$ and $f_i^{(2)}(z,t)=0$, $\gamma^{(2)}(z,t)=0$, consequently $u_i^{(1)}(x,t)=g_{ip}(x-\xi,t)$, $p^{(1)}(x,t)=g_{pp}(x-\xi,t)$, $t_i^{(1)}(x,t)=f_{ip}(x-\xi,t)$, $q^{(1)}(x,t)=f_{pp}(x-\xi,t)$. And $z\in V$, $x$ is a point on $S$.

Thus we immediately have the following Somigliana-type integral equation:

$$
c_{ij}(\xi)u_i(\xi,t)=\int_S[\dot{G}_{ij}(x-\xi,t)*t_i(x,t)+G_{pj}(x-\xi,t)*q(x,t)
$$

$$
-\dot{F}_{ij}(x-\xi,t)*u_i(x,t)-F_{pj}(x-\xi,t)*p(x,t)]\,\mathrm{d}S(x), \tag{153}
$$

$$
c(\xi)p(\xi,t)=\int_S[-\dot{g}_{ip}(x-\xi,t)*t_i(x,t)-g_{pp}(x-\xi,t)*q(x,t)
$$

$$
+\dot{f}_{ip}(x-\xi,t)*u_i(x,t)+f_{pp}(x-\xi,t)*p(x,t)]\,\mathrm{d}S(x). \tag{154}
$$

where $i, j = 1,2,3$ for three-dimensional; $i, j = 1,2$ for two-dimensional; $x$ is the source point and $\xi$ is the field point; the superposed dot stands for time derivative. In eqns (153), (154) we have introduced new tensor functions $G_{ij}(x-\xi,t)$, $G_{pj}(x-\xi,t)$, $F_{ij}(x-\xi,t)$, $F_{pj}(x-\xi,t)$ which are the displacement (Green's function), pressure, traction and normal flux kernels due to a point force in the $j$-direction with Heaviside step function in time along with $g_{ip}(x-\xi,t)g_{pp}(x-\xi,t)$, $f_{ip}(x-\xi,t)$, $f_{pp}(x-\xi,t)$ which are similar kernels but due to a point source with Dirac delta function in time. Notice that all the displacement and the pressure kernel $G_{ij}(x-\xi,t)$, $G_{pj}(x-\xi,t)$ are available in the current work and in Chen (1992, 1994), while $g_{ip}(x-\xi,t)$ and $g_{pp}(x-\xi,t)$ can be obtained from the corresponding $G_{ip}(x-\xi,t)$, $G_{pp}(x-\xi,t)$ in the present work and Chen (1992, 1994) by simply taking the time derivative. Obviously the remaining traction and flux kernel $F_{ij}(x-\xi,t)$, $F_{pj}(x-\xi,t)$, $f_{ip}(x-\xi,t)$, $f_{pp}(x-\xi,t)$ can be derived from the displacement and pressure kernel through the constitutive relationship presented in the previous section. Additionally, $c_{ij}(\xi)$ and $c(\xi)$ is a matrix of constants which equals to one for $\xi \in V-S$, to zero for $\xi \in V_c$, and to $\frac{1}{2}\delta_{ij}$ and $\frac{1}{2}$ for $\xi$ on a smooth surface $S$.

Equations (153, 154) now provide the displacement $u_j(\xi,t)$, $p(\xi,t)$ at a point $\xi \in V$ due to any adequate combinations of $t_i$, $u_i$, $q$, $p$ over the boundary $S$. However, at a point on the boundary we know only either of $u_i$ or $t_i$, and $p$ or $q$, therefore the rest must be sought for beforehand to evaluate the internal displacements, pressure, stresses and flux. The unprescribed boundary-values can be obtained by solving the boundary integral equation, which is obtained by taking the field point $\xi$ onto the boundary $S$. A physical interpretation of the above integral representation is as follows: the integrands in the equations can be viewed as sources whose strength must be adjusted until the correct boundary and initial conditions are reproduced for the problem in question.

## NUMERICAL RESULTS

Since the analytical expressions of the Green's function are extremely complicated, it's best to investigate the essential features of the problem graphically. The results were obtained by evaluation of the analytical solutions and the accuracy of the time domain solutions was established by comparing with an accurate numerical inversion of Laplace transform solutions presented in eqns (47), and all plots are presented nondimensionally.

### *Numerical results for limiting case*

Following the definition of eqn (14), the nondimensional material parameters for Berea sandstone (Yew and Jogi, 1978; Burridge and Vargas, 1979) are as follows: $\lambda^* = 0.1715$, $\mu^* = 0.3007$, $Q^* = 0.3742$, $\rho^* = 1.0$, $\rho_f^* = 0.4325$, $m^* = 2.3006$, $\kappa^* = 1.0$, $\alpha = 0.779$.

Figures 1–4 depict the two-dimensional fundamental solution components for the short time approximation. In the current problem, the applied force point (or fluid source point) is located at (0,0), the receiver is chosen at nondimensional coordinate (0.1,0.15). The nondimensional velocities of the three waves are approximately $c_p = 1.0$ (pressure wave or $P_1$ wave); $c_d = 0.368$ (diffusive wave or $P_2$ wave) and $c_s = 0.572$ (shear wave). Thus they arrive at the receiver, at $t_p = 0.1793$ (pressure wave), $t_s = 0.3151$ (shear wave), and $t_d = 0.49$ (diffusive wave). All three arrival times can be detected on Fig. 1(c) by sudden changes appearing in displacement due to line force, while only two arrival times corresponding to the first compressional wave ($P_1$ wave) and second compressional wave ($P_2$ wave) can be identified on Figs 2, 3 and 4 by sudden changes in displacement due to point source injection and by two pulses appearing in excessive pore fluid pressure due to point force or point source injection. The pulses take the form of $1/(\sqrt{t^2-r^2/c^2})$. In Figs 3 and 4, the dynamic curves display the jump discontinuity at the wave front and thereafter rapidly approach the corresponding quasi-static state, in turn, tend to steady state.

It is of some interest to examine the values of dissipation factors of three waves. They are $\eta_d = 0.2336$ (diffusive wave), $\eta_p = 0.00294$ (pressure wave), $\eta_s = 0.0192$ (shear wave). This confirms Biot's finding that the waves of the second kind (diffusive wave with large value of $\eta_d$) are highly attenuated and the waves of the first kind are true waves (pressure wave with negligible small value of $\eta_p$).

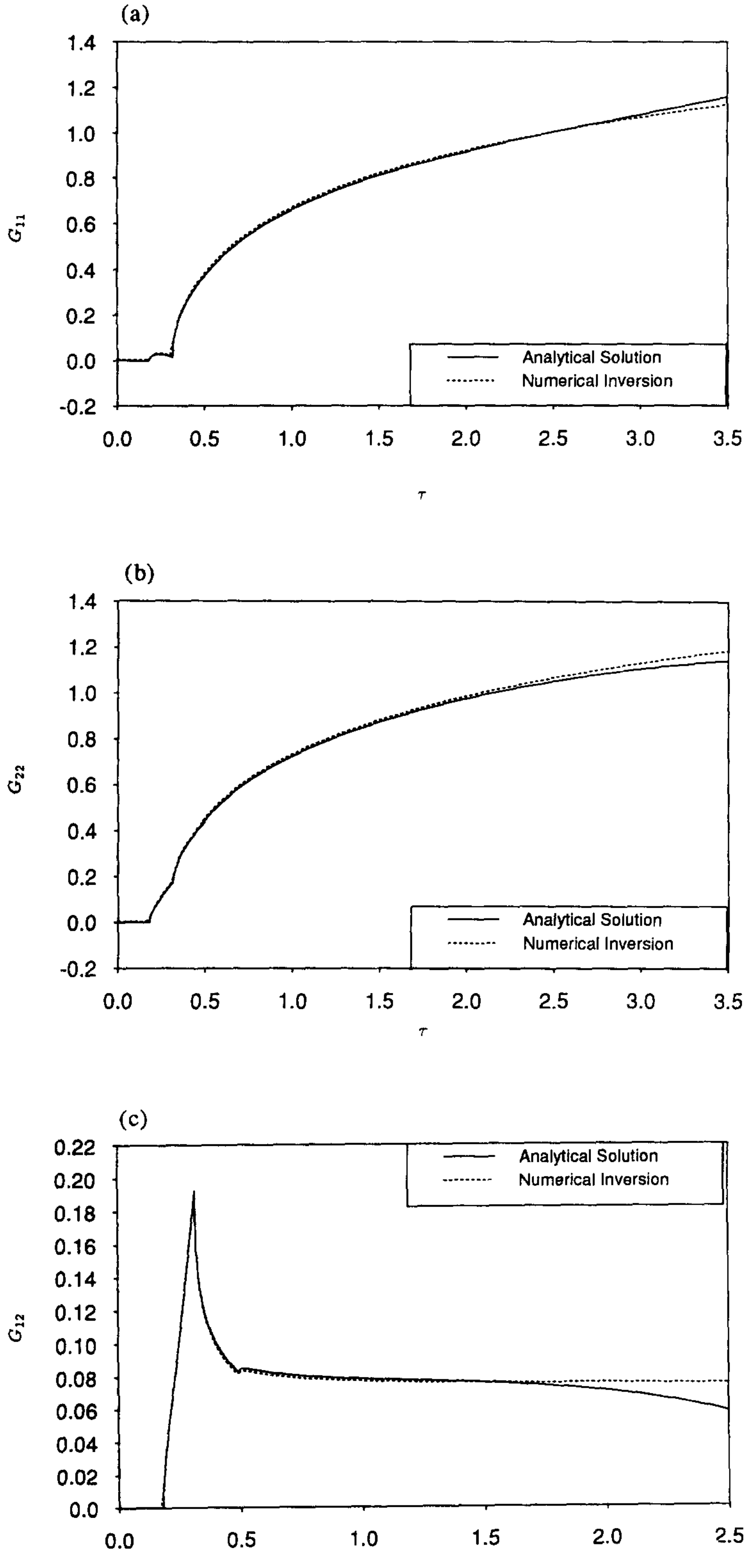


Fig. 1. Two-dimensional displacement time history at $\vec{\xi} = (0.1, 0.15)$ due to point force at (0, 0).

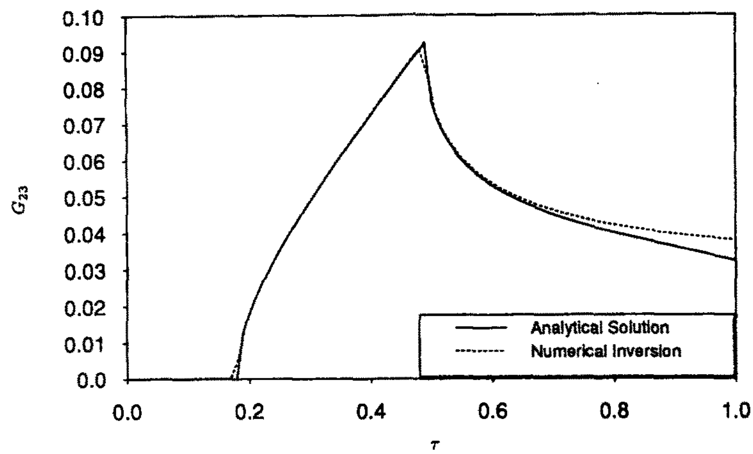


Fig. 2. Two-dimensional displacement time history at $\vec{\xi} = (0.1, 0.15)$ due to fluid injection at $(0, 0)$.

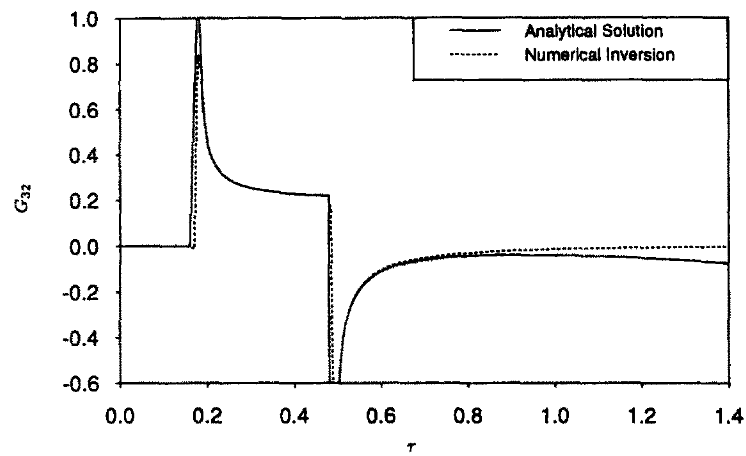


Fig. 3. Two-dimensional pressure time history at $\vec{\xi} = (0.1, 0.15)$ due to point force at $(0, 0)$.

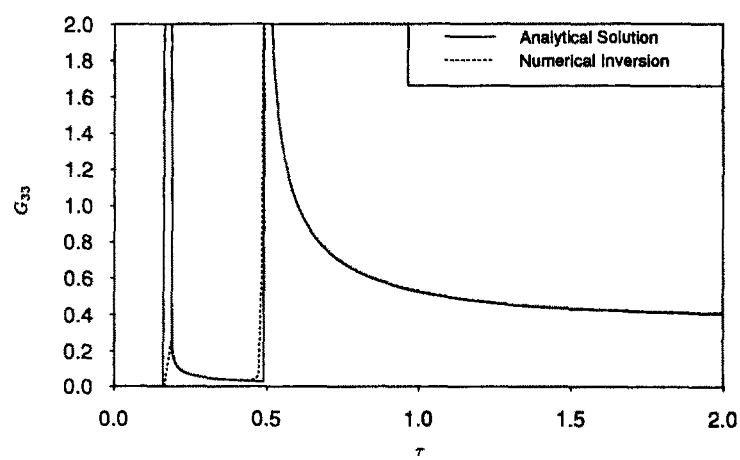


Fig. 4. Two-dimensional pressure time history at $\vec{\xi} = (0.1, 0.15)$ due to fluid injection at $(0, 0)$.

In order to examine the accuracy of the analytical solutions all the figures are plotted through a comparison with results from a numerical inversion of the Laplace transform solutions. The comparison demonstrates that the analytical solutions can capture the salient nature and characteristics of wave propagation in porous media near the arrival time. However, as predicted earlier, the accuracy of the analytical solutions deteriorates and deviates as time increases. To solve the dilemma of this short term solution, an entirely satisfactory transient fundamental solution for the general case has been developed in the previous section and is to be studied via plots in the next subsection.

*Numerical results for the general case*

The material parameters for Pecos sandstone (Yew and Jogi, 1978; Burridge and Vargas, 1979) are presented in nondimensional form as: $\lambda^* = 0.1286$, $\mu^* = 0.2746$, $Q^* = 0.4679$, $\rho^* = 1$, $\rho_f^* = 0.4399$, $m^* = 2.256$, $\kappa^* = 1$, $\alpha = 0.83$.

Figures 5–8 show the two-dimensional Green's function components for a general case. In the current problem the applied force point (or fluid source point) is located at (0, 0), the receiver is chosen at the nondimensional coordinate (1, 2). The nondimensional wave velocities are approximately $c_p = 1.0$ (pressure wave or $P_1$ wave); $c_d = 0.3918$ (diffusive wave or $P_2$ wave); $c_s = 0.5475$ (shear wave). In view of the above, the nondimensional time

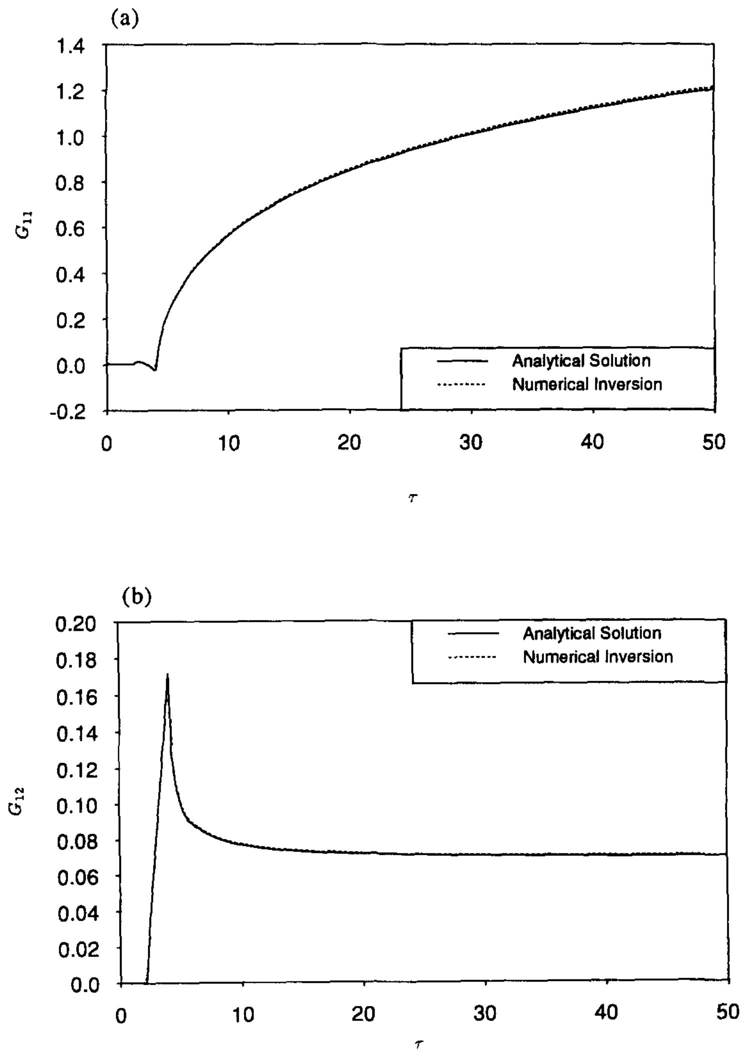


Fig. 5. Two-dimensional displacement time history at $\xi = (1, 2)$ due to point force at (0, 0).

required for the three waves to reach the receivers are $t_p = 2.236$ (pressure wave), $t_s = 4.0841$ (shear wave), and $t_d = 5.7065$ (diffusive wave). In all the figures, excellent agreement of analytical solution and numerical Laplace inversion is seen.

Figures 6–8 clearly demonstrate the existence of two wave fronts (pressure wave and diffusive wave); these fronts propagate with speeds $c_p$, and $c_d$, respectively.

An interesting feature is the presence of two pulses in those Green's functions which define the pressure due to line force or line source injection [Figs 7–8]. These pulses, in the form of $1/(\sqrt{t^2-r^2/c^2})$, are associated with the arrival of the two dilatational waves.

Figure 5 indicates the displacement component of Green's function due to the line force. Figure 5(b) shows that the solution contains three wave fronts, two dilatational waves and one rotational wave. They propagate with speeds $c_p$ (pressure wave), $c_s$ (shear wave) and $c_d$ (diffusive wave), respectively. Analytical solutions based on the general case have shown that the propagation of the fast wave (pressure wave) is characterized by nearly compatible deformations of the solid and fluid phases; very little viscous attenuation being induced with practically negligibly small dissipation factor $\eta_p \approx 0$. This strongly confirms Biot's (1956a, b, c) finding that the waves of the first kind are true waves, the dispersion is practically negligible. Therefore, the $P_1$ wave can be detected in the far field as well as the near field of the source. The propagation of the slow wave ($P_2$ wave) is characterized by

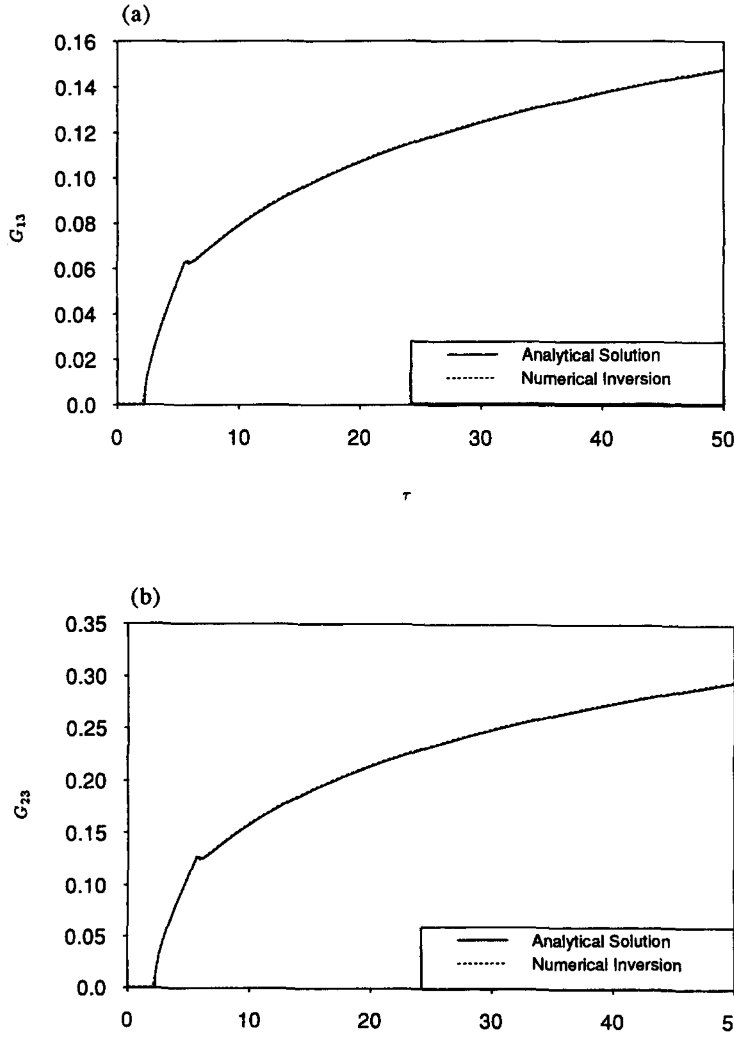


Fig. 6. Two-dimensional displacement time history at $\xi = (1, 2)$ due to fluid injection at $(0, 0)$.

the fluid and solid dilatations being nearly 180° out of phase and the propagation is strongly attenuated with a high dissipative factor, $\eta_d = 0.242$, and hence it can be detected only at a close proximity to the source. As the disturbance moves into the media, the diffusive wave front slows down and eventually disappears. The dissipative factor $\eta_s = 0.0199$ for shear wave is much smaller than diffusive wave. Obviously the attenuation enters through the inclusion of a damping term in the original equations, eqn (3) to the difference of solid and fluid velocities.

We turn, next, to the discussion of the interaction between the wave propagation and diffusion process. The fact that no diffusion takes place before the arrival of diffusive wave and it starts right away after its arrival is evident from Figs 7 and 8. This is primarily due to high viscous attenuation of diffusive wave which is in the nature of a diffusion process, and the propagation is closely analogous to heat conduction or related to consolidation.

These plots reveal a significant characteristic in the two-dimensional fundamental solutions, i.e. the response exhibits a tail immediately behind the wave front when plotted with respect to time. The reason for this phenomena is that in the two-dimensional case, the disturbance keeps reaching the receiver and the response continues to exist after the arrival of the first disturbance. This can be more clearly explained by the fact that the disturbance in two dimensions is uniformly distributed along the $x_3$-axis, thus signals continuously arrive at the receiver from all points along the $x_3$-axis.

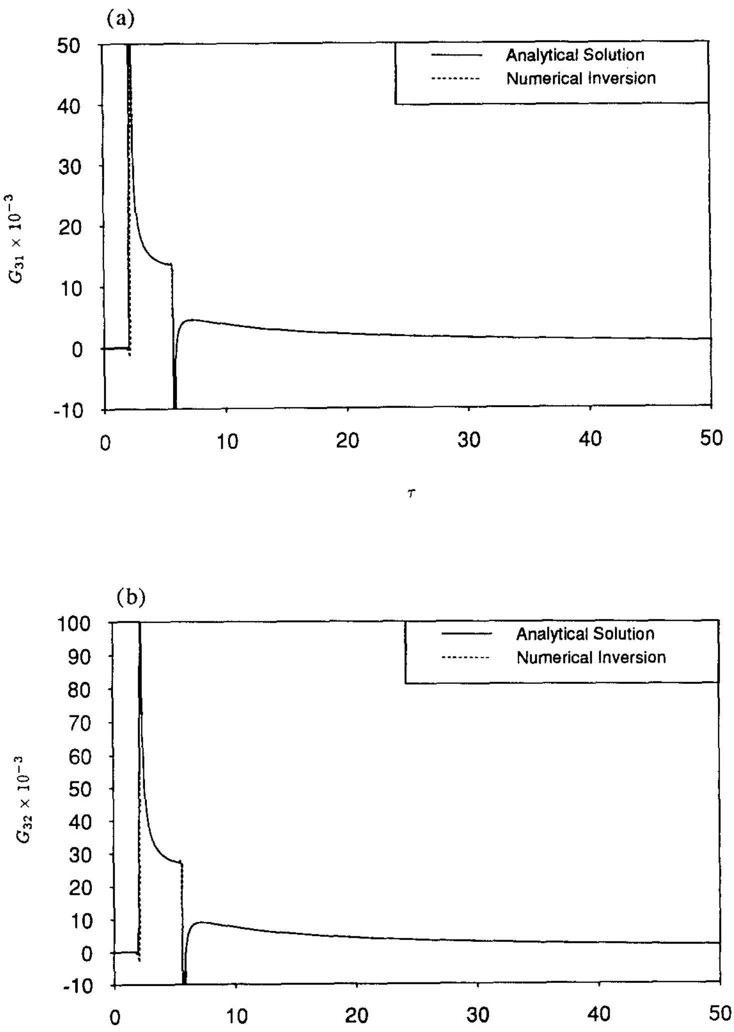


Fig. 7. Two-dimensional pressure time history at $\vec{\xi} = (1, 2)$ due to point force at $(0, 0)$.

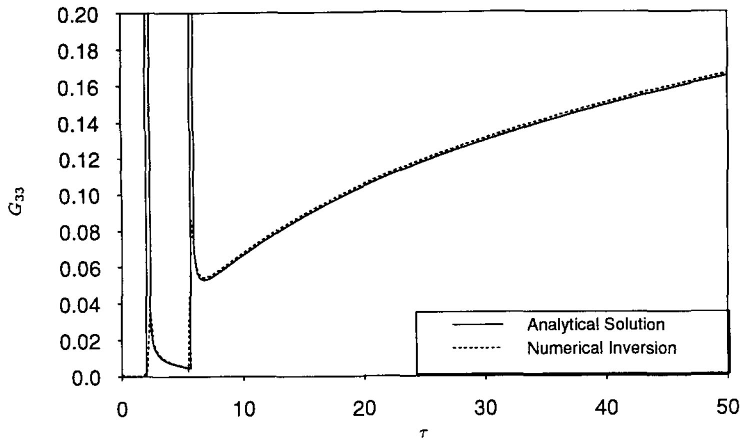


Fig. 8. Two-dimensional pressure time history at $\vec{\xi} = (1, 2)$ due to fluid injection at (0, 0).

## CONCLUSION

The closed-form transient Green's functions for the two-dimensional dynamic poroelasticity subject to Heaviside step loadings were obtained for both the limiting case and general case. The transient fundamental solutions clearly demonstrate that under the imposition of dynamic loadings there develop two dilatation waves and one rotational wave; each involve coupled motions of both the solid and fluid. The fast dilatational wave ($P_1$ wave), traveling at a speed $c_p$, controlled largely by the compressibility of the framework of grains and corresponds to the usual $P$ wave considered in seismology is characterized by solid and fluid motions that are in-phase. By contrast, the slow dilatational wave ($P_2$ wave), traveling at a speed $c_d$ close to (but usually less than) the fluid velocity, is characterized by out-of-phase motions of the solid and fluid, and it dissipates energy through diffusion of the fluid, therefore it attenuates very rapidly. The dissipative factors $\eta_d, \eta_s, \eta_p$ are largest for slow dilatational wave and much smaller for rotational wave, which travels with speed $c_s$, and extremely small for fast dilatational wave, respectively. This is an indication of the highly dissipative nature of slow dilatational wave, the smaller dissipative nature of transverse wave and practically negligible dissipative nature of the fast dilatational wave. The transient fundamental solutions also show that while the point forces generate all three kinds of waves in displacement fields, the contribution of transvertional wave in the pressure is zero and the scalar fluid source also does not generate any shear wave. Although the integrals appearing in some terms of these equations are not solved explicitly, the expressions are readily amenable to numerical computation. The availability of a satisfactory transient fundamental solution and an accurate transient boundary integral equation for full dynamic poroelasticity enables the development of an efficient time domain BEM to solve more practical problems related to dynamic nonlinear poroelasticity, dynamic soil–structure interaction, seismic wave scattering, earthquake engineering, acoustics and biomechanics.

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